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CONTENTS

OPTIMAL CONTROL OF A NONLINEAR SYSTEM

Sun Tszyan'

(Moscow)

(Translated from: Avtomatika i Telemekhanika, Vol. 21, No. 1, pp. 3-14, January, 1960)
Original article submitted May 26, 1959

The paper considers transient responses, optimal as concerns speed of response, in control systems containing a rotary amplifier and a dc motor. The motor's excitation voltage is considered as a second independent control parameter. Methods are suggested for the synthesis of an optimal controlling device. An example is given of the synthesis for simplified second- and third-order systems.

Electrical control and regulation systems containing rotary amplifiers and dc motors have found wide application. The various systems in use today are ordinarily constructed on the basis of the linear theory. Control of the motor is effected on the side of the rotary generator's control winding, wherein the controlling signals are proportional to a combination of the error (the discrepancy between input and output) and its derivatives. The motor's excitation voltage is ordinarily fixed. These systems can also contain other connections, introduced for the purpose of compounding or compensation. In all these systems, the signal applied to the rotary generator's control winding is the sole control parameter. However, a second possible control parameter – the motor's excitation voltage – exists in these systems. With a variable motor excitation voltage, the system can manifest better quality indications, for example, higher speeds of response.

A system with two control parameters is nonlinear, as will become obvious in the sequel. The analysis and synthesis of such optimal systems are very difficult variational problems. Today, the problem of synthesizing such optimal systems can be solved on the basis of the work of L. S. Pontryagin, V. G. Boltyanskii, and R. V. Gamkrelidze [1-3]. In these works, the variational maximum principle is proven to be a necessary condition for optimality of control. However, in many cases, the maximum principle, in conjunction with other information about a system, permits one to solve completely the problem of synthesizing an optimal system.

1. Equation of System Motion

We consider the electrical control system containing a rotary amplifier whose block schematic is shown on Fig. 1. With the nomenclature used on this figure, we can write the following system of equations:

$$\frac{dX_1}{dt} = \zeta\Omega, \qquad J\frac{d\Omega}{dt} = k_3 I_a I_e + M_{st},$$
$$T_1\frac{dI_c}{dt} + I_c = k_1 U_1, \qquad T_2\frac{dI_e}{dt} + I_e = k_2 U_2,$$
$$T_3\frac{dE_g}{dt} + E_g = k_4 I_c, \qquad E_g - c\Omega I_e = I_a R_a. \tag{1}$$

Here, ζ, k_1, k_2, k_3, and k_4 are constant gains, T_1, T_2, and T_3 are the time constants of the rotary amplifier's control winding, in the motor's excitation winding and in the transverse circuit of the rotary amplifier, respectively. In our further transformations of system of equations (1) we shall made the following assumptions: the load's static

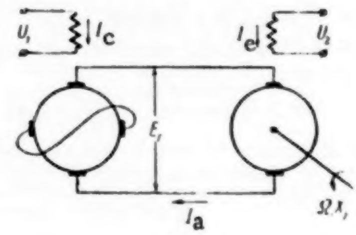

Fig. 1.

torque is small in comparison with the load's inertia; time constant T_1 of the rotary amplifier's control winding is small in comparison with T_3. System (1) will then, after some transformation and after conversion to dimensionless units, have the following form:

$$\frac{dx}{dt} = \zeta_0\omega, \qquad \frac{d\omega}{dt} = \frac{\eta}{1-\mu} i_e e_g - \omega i_e^2,$$
$$\frac{de_g}{dt} = -\beta_1 e_g + \beta_1 u_1, \qquad \frac{di_e}{dt} = -\beta_2 i_e + \beta_2 u_2. \tag{2}$$

In these equations,

$$x = \frac{X_1}{X_{1\,max}}, \quad e_g = \frac{E_g}{E_{g\,max}}, \quad i_e = \frac{I_e}{I_{e\,max}},$$

$$i_a = \frac{I_a}{I_{a\,nom}}, \quad \omega = \frac{\Omega}{\Omega_{nom}}, \quad u_1 = \frac{U_1}{U_{1\,max}},$$

$$u_2 = \frac{U_2}{U_{2\,max}}, \quad \beta_1 = \frac{T_{em}}{T_3}, \quad \beta_2 = \frac{T_{em}}{T_2},$$

$$T_{em} = \frac{JR_a}{k_3 C I_{e\,max}^2}, \quad \eta = \frac{E_{g\,max}}{E_{g\,nom}} > 1,$$

$$\mu = \frac{I_{a\,nom} R_a}{E_{g\,nom}} < 1.$$

The time $\underline{t}$ in system (2) of equations is also dimensionless; as the basic quantity we take the electromechanical time constant T_{em}. For convenience, we retain the previous notation for time.

In actual systems there are four essential limitations: limitations on the control and excitation voltages u_1 and u_2, limitations on armature current, and limitations on the motor's rotational speed. The last two are limitations on coordinates of the system. The analysis and synthesis of optimal systems when limitations on coordinates are taken into account are complicated mathematical problems requiring special study. However, the coordinate limitations in practice are quite simple. In many cases, quite simple considerations permit one to find the optimal processes with coordinate limitations taken into account [5]. In this paper we shall only consider limitations on the controlling parameters, as the case most studied from the point of view of the variational calculus. In relative units, the limitations imposed on u_1 and u_2 are expressed by the following inequalities:

$$|u_1| \leqslant 1, \quad 0 < \lambda \leqslant u_2 \leqslant 1. \tag{3}$$

The lower limit for the motor excitation voltage λ is defined by the condition that the motor not enter a separation mode of operation for small static loads.

We shall investigate the optimal transient responses in the processing of original errors. In this case, the magnitude of an original error does not effect the structure of the optimal controlling portion. When the input stimulus varies linearly in time, the problem is solved analogously, but a more complicated structure (design) is obtained since, in this case, the velocity at which the input stimulus varies will enter into the equation of motion. Let the input stimulus be $g(t) = A_0$, where A_0 is an arbitrary constant. We introduce new coordinates, related to the old coordinates in the following way:

$$x_1 = A_0 - x, \quad x_2 = \frac{dx_1}{dt} = -\zeta_0\omega,$$
$$x_3 = -e_g, \quad x_4 = i_e. \tag{4}$$

By substituting the new coordinates in Eqs. (2), we get the definitive system of equations:

$$\frac{dx_1}{dt} = x_2, \quad \frac{dx_2}{dt} = \alpha x_3 x_4 - x_2 x_4^2,$$
$$\frac{dx_3}{dt} = -\beta_1 x_3 - \beta_1 u_1, \quad \frac{dx_4}{dt} = -\beta_2 x_4 + \beta_2 u_2. \tag{5}$$

Expressions (3) for the limitations on u_1 and u_2 also hold for equations in the new coordinates.

2. Optimal Processes and Optimal Equations

Let the initial state of the system be given, i.e., at the initial moment of time let the four coordinates of system of equations (5) be given. It is required to choose controls $u_1(t)$ and $u_2(t)$ which will take the system to the null state in minimal time. We first consider the phase space X of system (5). Each point of this space, called a representative point, defines one state of the system. The system's null state is characterized by the point with coordinates $\{0, 0, 0, \gamma\}$, where $\gamma (0 < \lambda < \gamma < 1)$ is some fixed number. The system's motion will be considered as a trajectory of the representative point $\{x_1, x_2, x_3, x_4\}$ in phase space X. The general point in X is denoted by the letter x.

Limitations (3) form a closed rectangle U on the control plane, whose coordinates axes are u_1 and u_2. The control vector with coordinates u_1 and u_2 we shall denote by the letter $\underline{u}$. A vector function u(t) is called admissible if all its values lie in U. In the sequel we shall assume that u(t) is an admissible piecewise-smooth function with no more than a finite number of first-order discontinuities. Each such fixed control uniquely defines one continuous trajectory which passes through the given initial point in phase space.

With this notation, the system of equations in (5) can be replaced by one vector equation

$$\frac{dx}{dt} = f(x) + Bu. \tag{6}$$

Here, B is the rectangular matrix

$$B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -\beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix}.$$

The optimal control problem can be formulated as follows: In phase space X the point x(0) is given, and it is required to find an admissible control $u(t) = \{u_1(t), u_2(t)\}$ which will carry point x(0) over into null point $\xi_0 = \{0, 0, 0, \gamma\}$ in minimal time.

In accordance with the theorems of the maximum principle [3], we set up the system of differential equations which is adjoint to system (5) with the new variables $\psi = \{\psi_1, \psi_2, \psi_3, \psi_4\}$:

$$\frac{d\psi_1}{dt} = 0, \quad \frac{d\psi_2}{dt} = -\psi_1 + x_4^2\psi_2,$$
$$\frac{d\psi_3}{dt} = -\alpha x_4\psi_2 + \beta_1\psi_3, \tag{7}$$
$$\frac{d\psi_4}{dt} = (2x_2x_4 - \alpha x_3)\psi_2 + \beta_2\psi_4.$$

Here, ψ is a covariant vector whose components are computed by the usual formula

$$\frac{d\psi_i}{dt} = -\sum_{\alpha=1}^{n} \frac{\partial f_\alpha}{\partial x_i}\psi_\alpha \quad (i = 1, 2, 3, 4).$$

We now set up the Hamiltonian function, which is here the scalar product of the vector ψ by the velocity vector dx/dt. We denote by H the Hamiltonian function:

$$H(x, \psi, u) = \left(\psi, \frac{dx}{dt}\right)^* = \left(\psi, f(x) + Bu\right) =$$
$$= (\psi, f(x)) + (\psi, Bu). \tag{8}$$

* We shall use parentheses to denote the scalar product of two vectors.

With this notation, we formulate the necessary condition for the optimality of control u(t) in the following way [3]. Let u(t) be an optimal control and x(t) the corresponding optimal trajectory. There then exists a nonzero continuous vector function $\psi(t)$ such that

$$H(\psi(0),\ x(0),\ u(0)) \geqslant 0,$$

the functions $\psi(t)$, x(t), and u(t) satisfy systems (5) and (7) of equations, so that along the trajectory, at any moment of time, the Hamiltonian function is a maximum with respect to u, i.e.,

$$H(\psi(x),\ x(t),\ u(t) = \sup_{u \in U} (\psi(t),\ f(t) + Bu)$$

and

$$H(\psi(t),\ x(t),\ u(t)) = \text{const} \geqslant 0. \tag{9}$$

We note that the quantity in the first parentheses in the rightmost member of (8) does not depend on u. Consequently, the expression in the second parentheses attains a maximum simultaneously with Hamiltonian function H. The necessary condition for optimality leads to the condition

$$(\psi(t),\ Bu(t)) = \sup_{u \in U} (\psi(t),\ Bu). \tag{10}$$

We now write (10) in expanded form:

$$(\psi,\ Bu) = -\psi_3(t)\,\beta_1 u_1 + \psi_4(t)\,\beta_2 u_2.$$

If an optimal control exists, it is easily seen that it is defined by the following relationships:

$$u_1(t) = -\operatorname{sign} \psi_3(t),$$

$$u_2(t) = \begin{cases} 1 \text{ for } & \psi_4(t) > 0, \\ \lambda \text{ for } & \psi_4(t) < 0. \end{cases} \tag{11}$$

This means that if the optimal control exists, it consists of a certain number of intervals, in each of which u_1 and u_2 assume one of their limiting values.

Controls u_1 and u_2 change their values by a jump each time $\psi_3(t)$ and $\psi_4(t)$ pass through zero. This is the state of affairs in all nonlinear systems where the controlling parameters enter linearly and if the corresponding components of the covariant vector do not vanish identically in any finite interval of time.

We now give the solution of system (7) of equations for the vector $\psi(t)$

$$\psi_1(t) = \psi_{10} = \text{const},$$

$$\psi_2(t) = e^{\int_0^t x_4^2(\tau)\,d\tau} \left[\psi_{20} - \psi_{10} \int_0^t e^{-\int_0^{\tau} x_4^2(s)\,ds}\, d\tau\right],$$

$$\psi_3(t) = e^{\beta_1 t}\left[\psi_{30} - \alpha \int_0^t x_4(\tau)\,\psi_2(\tau)\,e^{-\beta_1 \tau} d\tau\right],$$

$$\psi_4(t) = e^{\beta_1 \tau}\left[\psi_{40} + 2\int_0^t (x_2(\tau)\,x_4(\tau) - \alpha x_3(\tau))e^{-\beta_1 \tau} d\tau\right]. \tag{12}$$

It is clear from (12) that $\psi_2(t)$ goes through zero no more than once. Consequently, if $x_4(t)$ does not change sign, $\psi_3(t)$ goes through zero not more than twice and, in the general case, $u_1(t)$ consists of three intervals.

These consequences are useful for the solution of the synthesis problem. To be assured of the possibility of constructing an optimal system, it is nice to know the following theorem on the existence of optimal controls: For each point of the phase space given by (5), there exists an optimal control which takes this point to the point $\xi_0 = \{0, 0, 0, \gamma\}$ in minimal time.

The proof of this theorem for the system of (5) can easily be carried through by modifying the well-known scheme of the proof of the similar theorem for linear systems which has been given by R. V. Gamkrelidze [2]. This may be done because the nonlinearity in (5) enters only into the coefficients of the corresponding equations.†

Concrete analysis shows that one can expect the optimal control in our case to be unique.‡ This means that, for a given point of phase space, there exists a unique control which takes this point over into point ξ_0 in minimal time. We shall make use in the sequel of the existence and uniqueness theorems for optimal controls.

3. Some Properties of the Field of Optimal Trajectories

In this section we give a brief presentation of some of the properties of optimal trajectories. As was stated earlier, we can assume existence and uniqueness of the optimal controls for the system being studied, i.e., for any point of phase space there exists a unique optimal trajectory joining the given point to the null point ξ_0. To each point of phase space there corresponds just one duration of the optimal transient response. The time T is a single-valued function of the phase-space points. We now consider the set S_T of points with the same duration of the optimal transient response. All these points lie on some hypersurface with equation

$$T(x_1,\ x_2,\ x_3,\ x_4) = C. \tag{13}$$

The entire space is filled by the optimal trajectories. In the direction of point ξ_0 these trajectories can only converge, not diverge. On each optimal trajectory from a remote point there lies only one point which is in the set S_T.

† The theorem on the existence of optimal controls is proved in [7] for nonlinear nonstationary systems with linear controls and for certain additional limitations.

‡ What we have in mind here are the concrete constructions of optimal processes on planes and in three-dimensional space. So far as the author knows, the literature contains no proof of the existence theorem for an arbitrary nonlinear system.

It may be proved** that the point set S_T is a closed convex hypersurface. Moreover, this hypersurface is everywhere continuous and almost everywhere differentiable, i.e., except for a set of points of "area" (measure) zero, there is a tangent plane at every point of S_T. Following A. Ya. Lerner [5], we call S_T an isochrone. All points inside S_T have a duration of the transient response less than T. An isochrone continuously widens with increasing time. As $T \to \infty$, S_T fills the entire finite phase space.

The function $T(x_1, x_2, x_3, x_4)$ forms a scalar potential field in phase space. Equation (13) defines an equipotential hypersurface. Since hypersurface S_T is differentiable almost everywhere, field T generates a vector gradient field, which is the field of external normals to hypersurface (13). One may easily remark the relationship between the field of external normals and the covariant vectors ψ, which is noted in L. S. Pontryagin's maximum principle. It turns out that the vector ψ forms a vector field in phase space X, with the exception of the set of points of the space where the hypersurface S_T is not differentiable. This can be written in the form of a vector equation

$$\operatorname{grad} T(x_1, x_2, x_3, x_4) = \\ = \sigma(x_1, x_2, x_3, x_4)\, \psi(x_1, x_2, x_3, x_4), \tag{14}$$

where σ is some function of the points of space X.

Equation (14) gives rise to several partial differential equations which, in some of the simplest cases, can be integrated, giving the function T in explicit form. It may also be shown that there is a continuous dependence of the optimal control on the original position of the point,†† i.e., points which are close in phase space have "close" optimal controls. This fact was established for linear systems by F. M. Kirillova in [6].

4. The Synthesis of an Optimal Controlling Device

As is well known, the task of synthesis reduces to the finding of the optimal control in the form of a function of the points of phase space $u_i(x_1, x_2, x_3, x_4)$ $(i = 1, 2)$ or, what amounts to the same thing, to finding the switching hypersurface for the controls u_1 and u_2 since, according to expression (11), the optimal control has the form of a relay function. If the optimal control functions cannot be found exactly by analytic means, then the switching hypersurface can be found approximately. Optimal trajectories always consist of a finite number of segments of curves of four families, corresponding to the controls $u_1 = \pm 1$ and $u_2 = 1, \lambda$. With this, the entire space is divided into regions, in each of which u_1 and u_2 have definite values. The boundaries of these regions are also the switching hypersurface. Two methods can be proposed for finding the switching hypersurfaces approximately. In the first method, one sets up, on an analog computer, a model whose time runs in the reverse direction of system (5). Then, all the optimal trajectories leaving point ξ_0 will have the reverse direction. We also construct the conjugate model in accordance with system of equations (7), but with time t replaced by $-t$. We then connect these two models as in Fig. 2. Since each initial condition $\psi(0)$ defines one optimal trajectory, we can, by giving various values of $\psi(0)$, obtain a set of optimal trajectories. With this, the pulse generator sends a command at the moments of switching of u_1 and u_2 which causes the recorder to write the values $\{x_1, x_2, x_3, x_4\}$. Thanks to the linearity of Eqs. (7) in ψ_i $(i = 1, 2, 3, 4)$, $\psi(0)$ can be chosen so as to satisfy the condition:

$$\psi_{10}^2 + \psi_{20}^2 + \psi_{30}^2 + \psi_{40}^2 = 1.$$

For each trajectory we record the times of switching and the corresponding system coordinates at the moment of switching. By continuing this for a sufficient number of times, we obtain an approximate representation of the switching hypersurface in the form of a table. Using the tabular data, one can carry out the synthesis of the optimal controlling device by the method cited in [9].

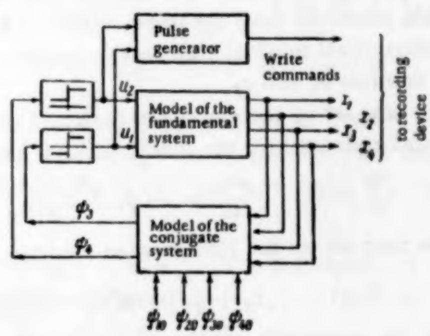

Fig. 2

If it is impossible to measure the system's parameters with sufficient accuracy, it is then advantageous to use another method for the determination of the optimal control function, wherein one uses the relationship between the gradient field of the duration of the optimal transient response and the vector field of the function $\psi(x_1, x_2, x_3, x_4)$. The isochrones for various durations of the optimal transient response can be constructed approximately by means of a machine of the type of the automatic optimizer which was developed under the direction of A. A. Fel'dbaum at the Institute of Automation and Remote Control of the Academy of Sciences, USSR. The algorithm for solving this problem on the automatic

** The proof of the following assertion, which we omit here, depends essentially on the specific features of nonlinear system (5), and is therefore of a concrete, nongeneral character. It is different from the well-known proof for linear systems.

†† A continuous dependence here has the mathematical meaning of continuity of the norm. If $u^1(t)$ and $u^k(t)$ are two optimal controls which take points ξ^1 and ξ^k to the null point then, as ξ^k tends to ξ^1, the integral $\int_0^T [u^1(t) - u^k(t)]^2\, dt$ tends to zero, where $T = \max(T_1, T_k)$.

optimizer was developed by R. L. Stakhovskii. By constructing the isochrone hypersurfaces we can, using Eqs. (14), determine the distribution of the signs of the functions $\psi_3(x_1, x_2, x_3, x_4)$ and $\psi_4(x_1, x_2, x_3, x_4)$, and thereby find the switching hypersurface.

For certain limiting cases, when the order of Eqs. (5) can be lowered, the synthesis problem can also be solved analytically. We give below the results of synthesis for two limiting cases: for the case when the generator's time constant and the motor's excitation winding time constant are negligibly small in comparison with the electromechanical constant, and for the case when only the second of these can be neglected. In the first case, we can substitute $\beta_1 = \infty$ and $\beta_2 = \infty$ in system of equations (5). The equations will then have the form:

$$\frac{dx_1}{dt} = x_2, \qquad \frac{dx_2}{dt} = -u_2^2 x_2 - \alpha u_1 u_2. \qquad (15)$$

As before, the limitations imposed on the system have the form given in (3). For this limiting case, the maximum principle turns out to be sufficient for the determination of all optimal trajectories and the switching lines of controls u_1 and u_2.

We now set up the conjugate (adjoint) system of differential equations for the functions $\psi_1(t)$ and $\psi_2(t)$:

$$\frac{d\psi_1}{dt} = 0, \qquad \frac{d\psi_2}{dt} = -\psi_1 + u_2^2\psi_2. \qquad (16)$$

We then set up the Hamiltonian function:

$$H(x, \psi, u) = \psi_1 x_2 + \psi_2(-u_2^2 x_2 - \alpha u_1 u_2). \qquad (17)$$

For the essentially nonlinear system (15), it is not known, in general, if there is an optimal control for any arbitrary initial conditions and, if there is one, whether or not it is unique. But, since system of equations (15) is a limiting case of system (5), one may expect that system (15) also possesses these properties. The results of the synthesis substantiate this assumption.

We now write the solution of system of equations (16):

$$\psi_1(t) = \psi_{10} = \text{const},$$

$$\psi_2(t) = e^{\int_0^t u_2^2(\tau)\,d\tau}\left[\psi_{20} - \int_0^t \psi_{10} e^{-\int_0^\tau u_2^2(s)\,ds}\,d\tau\right]. \qquad (18)$$

We notice that $\psi_2(t)$ cannot change sign more than once. We now turn our attention to the Hamiltonian function (17). Within the parentheses there, α is a positive number and u_2 is always positive. Then, in accordance with the requirement of the maximum principle,

$$u_1 = -\operatorname{sign}\psi_2(t). \qquad (19)$$

To elicit the optimal law for control $u_2(t)$, we rewrite the Hamiltonian function in another form:

$$H(x, \psi, u) = \psi_1 x_2 + \psi_2\left[-x_2\left(u_2 + \frac{\alpha u_1}{2x_2}\right)^2 + \frac{\alpha^2 u_1^2}{4x_2}\right]. \qquad (20)$$

Let us consider separately the two possible cases.

a) $\psi_2(t) < 0$. In this case, $u_1 = 1$. For $x_2 > 0$, u_2 must assume its greatest value, equal to unity. On the lower half-plane, where $x_2 < 0$, the optimal control u_2 is defined as follows:

$$u_2 = \begin{cases} 1, & \text{if } \left|\dfrac{\alpha}{2x_2}\right| \geqslant 1, \\ \dfrac{\alpha}{2x_2}, & \text{if } \lambda \leqslant \left|\dfrac{\alpha}{2x_2}\right| \leqslant 1, \\ \lambda, & \text{if } \left|\dfrac{\alpha}{2x_2}\right| \leqslant \lambda. \end{cases} \qquad (21)$$

b) $\psi_2(t) > 0$, $u_1 = -1$. On the upper half-plane, where $u_2 > 0$, the optimal control u_2 is defined by the expression

$$u_2 = \begin{cases} 1 & \text{for } \dfrac{\alpha}{2x_2} \geqslant 1, \\ \dfrac{\alpha}{2x_2} & \text{for } \lambda \leqslant \dfrac{\alpha}{2x_2} \leqslant 1, \\ \lambda & \text{for } \dfrac{\alpha}{2x_2} \leqslant 1. \end{cases}$$

On the lower half-plane, $u_2 = 1$.

It is thus clear that in the general case u_1 consists of two intervals, in each of which u_1 assumes the value 1 or −1. Control u_2, which is the motor excitation, consists of four intervals. In one of these intervals, u_2 continuously sweeps through the segment $[\lambda, 1]$. On the succeeding segment u_2 can only assume its maximum value, 1, since otherwise the function H would not be a maximum at the endpoint, where $x_1 = x_2 = 0$. It may be noticed that, at the origin of coordinates, only two curves, from the different half-planes, pass. These two curves comprise the final segments of all optimal trajectories. We denote them by L_1 and L_1' (Fig. 3). On line L_1 control $u_1 = -1$ and on line L_1', control $u_1 = 1$. Analogous relationships lead us to the conclusion that ψ_2 changes sign on lines L_1 and L_1'. Generalizing what has been said, we conclude that the phase plane is divided into six regions (Fig. 3). Lines L_1 and L_1' divide the phase plane into two half-planes; in the left half-plane $u_1 = -1$ and, in the right one, $u_1 = 1$. Moreover, there is a strip in each half-plane, bounded by the lines Γ_1 and Γ_2 in the left half-plane and the lines Γ_1' and Γ_2' in the right one. If the initial point x(0) is in the third quadrant then the optimal trajectory rises upward with controls $u_1 = -1$ and $u_2 = 1$. Meeting line Γ_1, u_2 begins to vary in accordance with the law $u_2 = \alpha / 2x_2$. On leaving this strip, u_2 takes its minimum value, λ; when the trajectory meets line L_1', u_1 changes sign and u_2 jumps to its maximum value

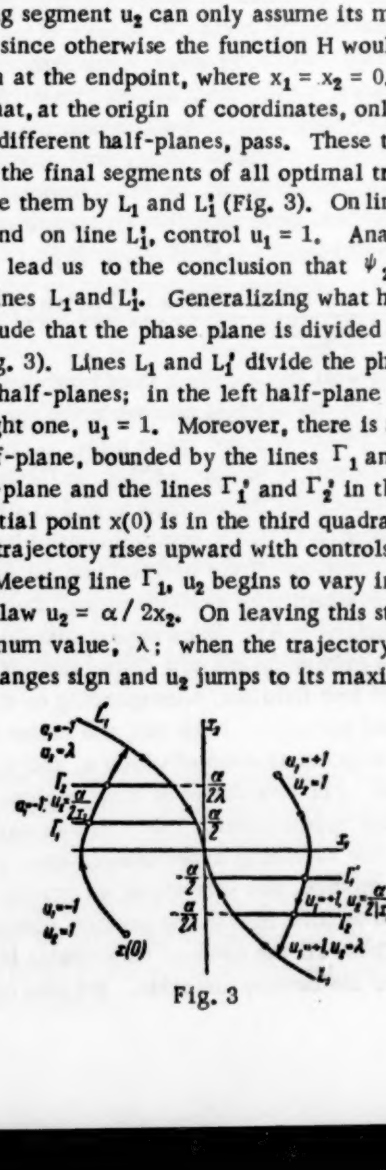

Fig. 3

after which the trajectory follows line L_1' to the origin of coordinates. The transient response and the laws of change of the controls with time are shown on Fig. 4.

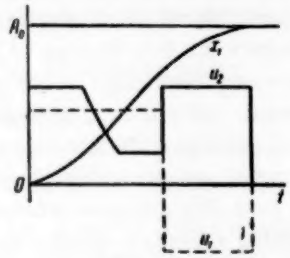

Fig. 4

We now consider the other case, when $\beta_2 = T_{em}/2 = \infty$, and β_1 is finite. In this case, we have the following system of differential equations

$$\frac{dx_1}{dt} = x_2, \quad \frac{dx_2}{dt} = \alpha x_3 u_2 - x_2 u_2^2,$$
$$\frac{dx_3}{dt} = -\beta_1 x_3 - \beta_1 u_1. \tag{22}$$

The adjoint system of equations for the vector ψ will be

$$\frac{d\psi_1}{dt} = 0, \quad \frac{d\psi_2}{dt} = -\psi_1 + u_2^2\psi_2,$$
$$\frac{d\psi_3}{dt} = 0 - \alpha u_2\psi_2 + \beta_1\psi_3. \tag{23}$$

The solution of this system is:

$$\psi_1(t) = \psi_{10} = \text{const},$$
$$\psi_2(t) = \left[-\psi_{10}\int_0^t e^{-\int_0^t u_2^2(s)\,ds}\,d\tau + \psi_{20}\right]e^{\int_0^t u_2^2(s)\,ds},$$
$$\psi_3(t) = \left[-\alpha\int_0^t u_2(\tau)\,\psi_2(\tau)\,e^{-\beta_1\tau}d\tau + \psi_{30}\right]e^{\beta_1 t}. \tag{24}$$

We can conclude from expressions (24) that, by virtue of the constancy of $\psi_1(t)$ along the given trajectory, $\psi_2(t)$ cannot change its sign more than once. Since $u_2(t)$ is assumed to be always positive, then $\psi_3(t)$, in the general case, does not change sign more than twice. The Hamiltonian function is

$$H(x,\ \psi,\ u) = \tag{25}$$
$$= \psi_1 x_2 + \psi_2(\alpha x_3 u_2 - x_2 u_2^2) + \psi_3(-\beta_1 x_3 - \beta_1 u_1).$$

For the Hamiltonian function to be maximal with respect to u_1, it is necessary that u_1 vary in accordance with the law

$$u_1 = -\operatorname{sign}\psi_3.$$

Thus, in general (in accordance with the remark made earlier), optimal control $u_1(t)$ consists of three intervals, in each of which u_1 assumes one of its boundary values. To determine the character of control $u_2(t)$, we rewrite the Hamiltonian function (25) in the following equivalent form:

$$H(x,\ \psi,\ u) = \psi_1 x_2 +$$
$$+ \psi_2\left[-x_2\left(u_2 - \frac{\alpha x_3}{2x_2}\right)^2 + \left(\frac{\alpha x_3}{2x_2}\right)^2 x_2\right] + \tag{25'}$$
$$+ \psi_3(-\beta_1 x_3 - \beta_1 u_1).$$

We consider separately the following cases.

1) $x_2 > 0$, $x_3 > 0$. When $\psi_2 < 0$, the Hamiltonian function H, according to (25'), attains a maximum for

$$u_2 = \begin{cases} 1, & \text{if } \dfrac{\alpha x_3}{2x_2} < \dfrac{1+\lambda}{2}, \\ \lambda, & \text{if } \dfrac{\alpha x_3}{2x_2} > \dfrac{1+\lambda}{2}. \end{cases}$$

When $\psi_2 > 0$, H is a maximum for

$$u_2 = \begin{cases} 1, & \text{if } \dfrac{\alpha x_3}{2x_2} \geqslant 1, \\ \dfrac{\alpha x_3}{2x_2}, & \text{if } \lambda \leqslant \dfrac{\alpha x_3}{2x_2} \leqslant 1, \\ \lambda, & \text{if } \dfrac{\alpha x_3}{2x_2} \leqslant \lambda. \end{cases}$$

2) $x_3 > 0$, $x_2 < 0$. In this case, the conditions for a maximum for H will be: $u_2 = \lambda$ for $\psi_2 < 0$ and $u_2 = 1$ for $\psi_2 > 0$.

3) $x_3 < 0$, $x_2 < 0$. The corresponding conditions are written as follows:

for $\psi_2 > 0$,

$$u_2 = \begin{cases} 1, & \text{if } \dfrac{\alpha x_3}{2x_2} < \dfrac{1+\lambda}{2}, \\ \lambda, & \text{if } \dfrac{\alpha x_3}{2x_2} > \dfrac{1+\lambda}{2}; \end{cases}$$

for $\psi_2 < 0$,

$$u_2 = \begin{cases} 1, & \text{if } \dfrac{\alpha x_3}{2x_2} \geqslant 1, \\ \dfrac{\alpha x_3}{2x_2}, & \text{if } \lambda \leqslant \dfrac{\alpha x_3}{2x_2} \leqslant 1, \\ \lambda, & \text{if } \dfrac{\alpha x_3}{2x_2} \leqslant \lambda; \end{cases}$$

4) $x_3 < 0$, $x_2 > 0$. In this region, H attains a maximum if

$$u_2 = \begin{cases} 1 & \text{for } \psi_2 < 0, \\ \lambda & \text{for } \psi_2 > 0. \end{cases}$$

On the basis of the discussion given, we can, in general, construct all the optimal trajectories in phase space. However, this necessitates a great deal of computational work. The structure of the switching surfaces for both controlling parameters is quite complicated.

The construction of these surfaces is an independent problem for investigation. Here, we speak only of certain features of the optimal transient responses in the system described by Eqs. (22).

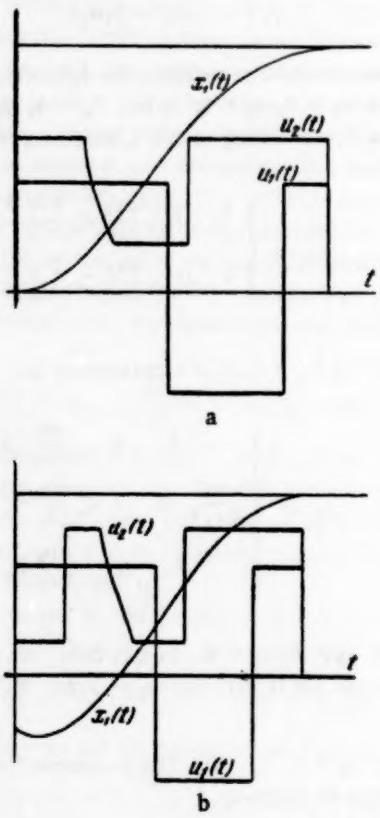

Fig. 5

Figure 5a gives the optimal transient response for zero initial conditions. Here, control $u_1(t)$, as it must, consists of three intervals. Control $u_2(t)$ has the same form as the corresponding control in the previous case. The difference amounts only to this: that the moment when u_2 switches from a minimum to a maximum does not coincide with the moment when u_1 is switched, but occurs later.

A completely different picture is obtained if the initial conditions are nonzero. As shown in Fig. 5b, for certain initial conditions, u_2 varies by a significantly more complicated law than in the case of the process with zero initial conditions.

CONCLUSIONS

The maximum principle of L. S. Pontryagin is a very powerful means for solving the problem of synthesizing control systems which give optimal speeds of response. Despite the fact that use of this principle leads only to necessary conditions for the optimality of controls, in certain cases it can, in conjunction with other information, provide sufficient information to permit the designing of systems with high speeds of response.

The generally accepted scheme for a control system containing a rotary amplifier with fixed motor excitation voltage is not the best one. It is more advantageous to introduce a variable excitation voltage whose law of variation can be exactly computed for a given system. By using the maximum principle, one can effect an exact synthesis of an optimal controlling device which makes the system best, in the sense of speed of response.

LITERATURE CITED

[1] V. G. Boltyanskii, R. V. Gamkrelidze, and L. S. Pontryagin, "On optimal process theory," Doklady Akad. Nauk SSSR 110, 1 (1956).

[2] R. V. Gamkrelidze, "Theory of optimally fast processes in linear systems," Izvest. Akad. Nauk SSSR, Ser. Matem. 22, 4 (1958).

[3] L. S. Pontryagin, "Optimal control processes," Usp. Matem. Nauk 14, 1, 85 (1959).

[4] A. A. Fel'dbaum, "On the synthesis of optimal systems by means of phase space," Avtomat. i Telemekh. 16, 2 (1955).

[5] A. Ya. Lerner, "On the limiting speed of response of automatic control systems," Avtomat. i Telemekh. 15, 6 (1954).

[6] F. M. Kirillova, "On the validity of the posing of one optimal control problem," Izvest. Vysshikh Ucheb. Zavedenii, Ser. Matem. 4 (1958).

[7] N. N. Krasovskii, "On a problem of optimal control of nonlinear systems," Prikl. Matem. i Mekh. 23, 2 (1959).

[8] R. Bellman, "On the bang-bang control processes," Quarterly Appl. Math. 14, 1 (April, 1956).

[9] Sun Tszyan', "Synthesis of the controlling portion of an optimally fast servo system," Avtomat. i Telemekh. 20, 3 (1959).‡‡

‡‡ See English translation.

THE REDUCTION OF NONLINEAR CONTROL SYSTEM EQUATIONS TO THEIR SIMPLEST FORM

E. N. Rozenvasser

(Leningrad)

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Formulas are given for linear transformations which take the equations of direct control systems to single nth-order differential equations. Use of this transformation allows one to shorten the computations required in carrying out certain calculations for control systems, for example, for determining the autooscillations in the case of piecewise-linear characteristics.

1. We shall consider the system of differential equations

$$\dot{x}_k = \sum_{\alpha=1}^{n} b_{k\alpha} x_\alpha + h_k f(\sigma,\ t) \quad (k = 1,\ 2, \ldots, n),$$

$$\sigma = \sum_{s=1}^{n} j_s x_s, \tag{1}$$

where $b_{k\alpha}$, h_k, and j_s are constants, and $f(\sigma, t)$ is some nonlinear function.

The characteristic polynomial of the linear portion of system (1) will be written in the form

$$D(\lambda) = |\lambda\delta_{k\alpha} - b_{k\alpha}| = \lambda^n + f_1\lambda^{n-1} + \ldots + f_n \tag{2}$$

($\delta_{k\alpha}$ is a Kronecker delta).

We now introduce the determinants $H_s(\lambda)$ ($s = 1, 2, \ldots, n$) which are obtained by replacing the sth column of determinant (2) by the column $\{H\}$ whose components are the number h_k. If we expand the determinants in powers of λ, we shall have

$$H_s(\lambda) = \sum_{k=1}^{n} h_{sk}\lambda^{n-k}. \tag{3}$$

We introduce the additional notation

$$M(\lambda) = \sum_{s=1}^{n} j_s H_s(\lambda) = \sum_{l=1}^{n} m_l \lambda^{n-l}. \tag{4}$$

We consider, in addition to (1), the system of equations

$$D(p)\,y = f(\sigma,\ t), \quad \sigma = M(p)\,y, \tag{5}$$

where D and M are the polynomials of (2) and (4) and p is the symbol for differentiation.

Let

$$y' = \frac{dy}{dt}, \quad y'' = \frac{d^2y}{dt^2}, \ldots, \quad y^{(n-1)} = \frac{d^{n-1}y}{dt^{n-1}}$$

and the determinant composed of the numbers h_{sk} be nonzero. Then, we have the relationship

$$x_i = \sum_{s=1}^{n} h_{is}\, y^{(n-s)}. \tag{6}$$

We now consider the columns

$$\{U_1\} = \{H\}, \tag{7}$$
$$\{U_2\} = B\{U_1\}, \ldots, \{U_n\} = B^{n-1}\{H\} = B\{U_{n-1}\},$$

where B is the matrix whose components are the numbers $b_{k\alpha}$.

We denote by Δ_n the determinants consisting, respectively, of the columns $\{U_1\}, \{U_2\}, \ldots, \{U_n\}$, and by Δ_n^s the algebraic complement of the element of the sth row and nth column of determinant Δ_n.

If we then solve system (6) for y and its derivatives, we get

$$y^{(n-k)} = \sum_{\alpha=1}^{n} m_{k\alpha} x_\alpha \tag{8}$$

where

$$m_{n\alpha} = \frac{\Delta_n^\alpha}{\Delta_n} \quad (\alpha = 1,\ 2, \ldots, n), \tag{9}$$

and the remaining quantities, m_{ps}, are defined by the recursion formulas

$$m_{ps} = \sum_{\alpha=1}^{n} m_{p+1,\alpha} b_{\alpha s}. \tag{10}$$

The derivations of formulas (6), (8), (9), and (10) are given in the Appendix.

2. For solving various control problems which relate to systems of the form of (1), it is convenient to make direct use of Eqs. (5) since, in every case, the computational work connected with the analysis of Eqs. (5) is simpler. As was noted earlier, Eqs. (5) can be written immediately if one knows the coefficients of the characteristic equation of linearized system (1) for $f(\sigma) = c\sigma$.

The subsequent transition from Eqs. (5) to the original system (1) is made by means of (6), and presents no difficulty.

In particular, let us consider the question of determining the autooscillations in system (1) in the case of a piecewise-linear characteristic. In system (1), let

$$f(\sigma, t) = f(\sigma) = a_i\sigma + b_i \quad \text{for} \quad \sigma_{i-1} < \sigma < \sigma_i, \tag{11}$$

where the σ_k are given. Then the corresponding system (5) will have the form:

$$\Big[D(p) - a_i M(p)\Big] y - b_i = 0$$

$$\text{for} \quad \sigma_{i-1} < M(p)\, y < \sigma_i. \tag{12}$$

With this, the search for continuous periodic solutions of system (1) reduces to the search for periodic solutions of Eqs. (12) which are continuous up to the (n-1)th derivative inclusive.

If we give ourselves the type of periodic mode of operation and then find the general solutions of the corresponding linear Eqs. (12) on each segment of variation of σ, we can, by matching up the solutions, construct a system of transcendental equations for the determination of the autooscillation period and the time of motion on each portion of the nonlinear characteristic. The matching-up method does not differ from the methodology presented in [1] as applied to system (1). However, the use of (12) significantly shortens the computational work involved in matching up and setting up the equations for the periods since, as is known [2], setting up the general solution of Eqs. (12) and its derivatives on each of the segments is much simpler than setting up the general solution of the original system (1).

One can also seek the periodic solutions of Eqs. (12) directly in the form of trigonometric series by the method, for example, proposed by L. A. Gusev [3].

However, in the given case, there exists an essential simplification. In [3], following M. A. Aizerman and F. R. Gantmakher [4], the author uses the derived equation of the form

$$D(p)\,\sigma - M(p)\, f(\sigma) = 0, \tag{13}$$

which was obtained from (1) by differentiation followed by elimination of unknowns. With this there are periodic solutions of Eq. (13) for definite conditions which relate the discontinuities of the derivatives of σ and $f(\sigma)$ at the moments of switching (saltus conditions). In [3], the saltus conditions were given in the form

$$\{\sigma\}_{q1} = U_q \{\sigma\}_{q2} - V_q, \tag{14}$$

where $\{\sigma\}_{q1}$ is a column with values $\sigma - m_n b_q/(f_n - a_q m_n)$, $\sigma, \ldots, \sigma^{(n-1)}$ at time $t_q - 0$, $\{\sigma\}_{q_2}$ is the same column at time $t_q + 0$, t_q is the time of the qth switching, U_q is a constant matrix, and V_q is a constant column.

By virtue of what has previously been said, the use of (13) as the derived equation allows one to replace the saltus conditions by the conditions of continuity of y, y', . . . , $y^{(n-1)}$ at the moments of switching. If, for this, we set up the conditions analogous to (14), we shall have

$$U_q = E, \quad V_q = \begin{vmatrix} \dfrac{b_q}{f_n - q_q m_n} - \dfrac{b_{q+1}}{f_n - a_{q+1} m_n} \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{vmatrix} \tag{15}$$

where E is the unit matrix.

For (15), Eqs. (27) of [3], which are used for the elimination of the arbitrary constants, are essentially simplified. Since the switchings occur for definite values of

$$\sigma_q = \sigma_{q\,\mathrm{giv}} \tag{16}$$

then, by eliminating the arbitrary constants, we arrive at equations for the periods which are analogous to Eqs. (30) of [3]:

$$\sigma_q(t_1, t_2, \ldots, t_h) = \sigma_{q\,\mathrm{giv}} \qquad (q = 1, 2, \ldots, h), \tag{17}$$

where $t_1, \ldots, t_h$ are the moments of switching.

If the times t_i are found, it is then easy to find y(t), y'(t), . . . , $y^{(n-1)}(t)$, after which, using (6), all the $x_i(t)$ are found. If Eq. (13) were used as the derived equation then, after finding the periodic solution $\sigma(t)$, the quantities $x_i(t)$ could only be found by integrating the corresponding system of linear inhomogeneous equations obtained from (1) by setting $\sigma = \sigma(t)$.

What has been presented here may also be used for finding the forced oscillations of system (1) for the case of piecewise-linear characteristics.

APPENDIX

Derivation of the Basic Formulas

For simplicity, we shall assume that the roots of equation $D(\lambda) = 0$: $\lambda = \lambda_\rho$ ($\rho = 1, 2, \ldots, n$) are simple, and also that, for each ρ, an m is found such that

$$H_m(\lambda_\rho) \neq 0.$$

Then, by introducing new variables z_ρ by the formulas

$$x_k = \sum_{\rho=1}^{n} \frac{H_k(\lambda_\rho)}{D'(\lambda_\rho)} z_\rho, \quad z_\rho = \frac{1}{H_m(\lambda_\rho)} \sum_{k=1}^{m} D_{km}(\lambda_\rho)\, x_k, \tag{18}$$

where $D_{km}(\lambda)$ is the algebraic complement of the kth row and mth column of determinant (2), we take system (1) to the canonical form of A. I. Lur'e:

$$\dot{z}_\rho = \lambda_\rho z_\rho + f(\sigma), \quad \sigma = \sum_{\rho=1}^{m} \frac{M(\lambda_\rho)}{D'(\lambda_\rho)} z_\rho. \tag{19}$$

On the other hand, system (5) also reduces to the form of (19) by means of the transformation:

$$z_\rho = \sum_{\alpha=1}^{n} B_{\alpha n}(\lambda_\rho)\, y^{(n-\alpha)}, \quad y^{(n-\alpha)} = \sum_{\rho=1}^{n} \frac{\lambda_\rho^{n-\alpha}}{D'(\lambda_\rho)} z_\rho, \tag{20}$$

where the $B_{\alpha n}(\lambda)$ are polynomials of the form

$$B_{1n} = 1, \quad B_{2n} = f_1 + \lambda,$$
$$B_{3n} = (f_1 + \lambda)\lambda + f_2, \ldots, B_{kn}\lambda = B_{k-1,n}\lambda + f_{k-1}. \tag{21}$$

By taking (18), (20), (21), and (3) into account, we can get

$$x_k = \sum_{l=1}^{n} p_{k\alpha} y^{(n-\alpha)}, \tag{22}$$

where the numbers $p_{k\alpha}$ are written in the form

$$p_{k\alpha} = \sum_{l=1}^{n} h_{kl}\pi_{l\alpha}, \tag{23}$$

where

$$\pi_{ls} = \sigma_{n-l+s-1} + f_1\sigma_{n-l+s-2} + \ldots + f_{s-1}\sigma_{n-l}$$
$$(1 \leqslant l \leqslant n,\ 1 \leqslant s \leqslant n), \tag{24}$$

and the quantities σ_k are defined by the relationships

$$\sigma_k = \sum_{\rho=1}^{n} \frac{\lambda_\rho^k}{D'(\lambda_\rho)}. \tag{25}$$

By taking into consideration the well-known equations for sums of the form of (25) [2]

$$\sigma_k = 0 \quad (k = 0, 1, \ldots, n-2),$$
$$\sigma_{n-1} = 1, \tag{26}$$

we obtain

$$\pi_{ls} = \begin{cases} 0 & \text{for } l > s, \\ 1 & \text{for } l = s. \end{cases} \tag{27}$$

Now, let $s \geq l + 1$. Then by virtue of (26) we have

$$f_{s-2}\sigma_{n-l-1} + \ldots + f_n\sigma_{s-l-1} = 0.$$

If we add this to (24), we find that

$$\pi_{ls} = \sigma_{n-l+s-1} + f_1\sigma_{n-l+s-2} + \ldots + f_n\sigma_{s-l-1}.$$

If we now take the identity

$$\frac{\lambda_\rho^{s-l-1}}{D'(\lambda_\rho)} D(\lambda_\rho) \equiv 0$$

and sum over all ρ, we get

$$\pi_{ls} = 0 \quad \text{for} \quad s > l. \tag{28}$$

We obtain (6) from (22), (23), (27), and (28). Formulas (9) can be obtained directly by solving system (6) for $y, y', \ldots, y^{(n-1)}$ and then carrying out certain simplifications. Relationships (10) are obtained in the following way. From (18) and (20) we have that

$$m_{k\alpha} = \sum_{\alpha=1}^{n} \frac{\lambda_\rho^{n-k} D_{\alpha m}(\lambda_\rho)}{D'(\lambda_\rho) H_m(\lambda_\rho)}. \tag{29}$$

We set up the sum

$$\sum_{\alpha=1}^{n} b_{\alpha s} m_{k\alpha} = \sum_{\alpha=1}^{n}\sum_{\rho=1}^{n} \frac{\lambda_\rho^{n-k} b_{\alpha s} D_{\alpha m}(\lambda_\rho)}{D'(\lambda_\rho) H_m(\lambda_\rho)}.$$

By reversing the order of summation and by taking into account the equation [5]

$$\sum_{\alpha=1}^{n} b_{\alpha\beta} D_{\alpha\gamma}(\lambda_\rho) = \lambda_\rho D_{\beta\gamma}(\lambda_\rho),$$

we obtain (10).

LITERATURE CITED

[1] M. A. Aizerman and F. R. Gantmakher, "On the determination of the periodic modes in nonlinear dynamic systems with piecewise-linear characteristics," Prikl. Matem. i Mekh. 5 (1956).

[2] A. I. Lur'e, Operational Calculus [in Russian] (Gostekhizdat, 1951).

[3] L. A. Gusev, "The determination of the periodic modes in automatic control systems which contain nonlinear elements with piecewise-linear characteristics," Avtomat. i Telemekh. 19, 10 (1958).*

[4] M. A. Aizerman, Lectures on Automatic Control Theory [in Russian] (Fizmatgiz, 1958).

[5] A. I. Lur'e, Some Nonlinear Problems of Automatic Control Theory [in Russian] (Gostekhizdat, 1951).

* See English translation.

THE RELATIONSHIP BETWEEN LINEAR SYSTEMS' TRANSIENT RESPONSES AND THEIR LAPLACE TRANSFORMS

N. S. Kochanov

(Leningrad)

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Methods are considered for the determination of discrete values of a linear system's transient response from its given Laplace transform, and for finding the Laplace transform from equally spaced given values of the transient response. These methods do not require that the roots of algebraic equations be computed.

Posing of the Problem

In solving the problem of analyzing the transient responses in linear systems with lumped parameters, great difficulty arises because of the necessity of calculating the roots of algebraic equations in order to make the transition from the transform of a time-domain function to the time-domain function itself.

The use of the methods of calculating the transient responses from frequency characteristics in such problems involves a significant amount of computation and graphical construction.

There is thus great interest in numerical methods of calculating transient responses which would allow one, in many cases, to compute quite simply discrete values of the transient response without determining the roots of the characteristic equation, and without graphical constructions.

Among these methods are the approximate operator method of G. S. Pospelov [1] and the similar methods discussed in the foreign literature [2, 3].

The practical use of these methods is significantly limited by the fact that, to obtain sufficiently high accuracy of the computations, one must choose a small discrete step, which leads to a corresponding increase in the amount of computational work, since the computations are carried out with constant steps.

With this, to determine discrete values of transient responses, one must carry out the expansion of several fractions in series.

The distinguishing features of the method proposed in the present work are the use of recursion formulas for the computations, and the use of a method for doubling the discrete step.

In the solution of the problem of synthesizing linear systems with lumped constants, the first question that arises is the determination of the transform of a given function of time. With this, due to the condition of physical realizability of such systems, the transform sought must be a rational fraction.

Thus, the second task of the present paper is the finding of methods for the determination of the Laplace transforms of functions of time in the form of exponential polynomials which approximate the given time-domain characteristics.

Transforms of Transient Responses

We shall consider impulsive responses, or regular parts of transient responses, which provide Laplace transforms in the form of rational fractions

$$\overline{F}(p)=\frac{b_0p^{n-1}+b_1p^{n-2}+\cdots+b_{n-1}}{p^n+a_1p^{n-1}+a_2p^{n-2}+\cdots+a_n}. \quad (1)$$

The expansion of these fractions in negative powers of p will have the form:

$$\overline{F}(p)=\frac{s_0}{p}+\frac{s_1}{p^2}+\frac{s_2}{p^3}+\cdots+\frac{s_k}{p^{k+1}}+\cdots \quad (2)$$

The coefficients s_k, which are the initial values of the function F(t) and its derivatives, satisfy the recursion relationship

$$s_{n+i}+a_1s_{n-1+i}+a_2s_{n-2+i}+\cdots+a_ns_i=0$$

$$(i=0,1,2,\ldots). \quad (3)$$

This relationship is obtained from the equation

$$p^n+a_1p^{n-1}+a_2p^{n-2}+\ldots+a_n=0 \quad (4)$$

by using the following systems of correspondences:

$$1=p^0\to s_0,\quad p\to s_1,\quad p^2\to s_2,\ldots,p^i\to s_i,\ldots \quad (5)$$

If the transient response is characterized by the system of discrete values $F_0, F_1, F_2, \ldots, F_{2n-1}, \ldots$, corresponding to equally spaced values of the argument t, then one can set up the series

$$\overline{F}(m)=\frac{F_0}{m}+\frac{F_1}{m^2}+\frac{F_2}{m^3}+\cdots+\frac{F_{2n-1}}{m^{2n}}+\cdots, \quad (6)$$

where $m = e^{p\tau}$ and τ is the discrete step.

By converting this series to a continued fraction [4], one can find a fraction which converges to this series:

$$\Phi(m) = \frac{B_0 m^{n-1} + B_1 m^{n-2} + \cdots + B_{n-1}}{m^n + A_1 m^{n-1} + A_2 m^{n-2} + \cdots + A_n}. \quad (7)$$

The expansion of this fraction by negative powers of m will coincide with series (6) up the term F_{2n-1}/m^{2n} inclusive, if 2n terms were used in converting the series to a continued fraction.

The following terms of this expansion will give the equally-spaced values of that nth-order exponential polynomial which, at the points $F_0, F_1, F_2, \ldots, F_{2n-1}$, assumes the given values.

If the equally spaced values $F_0, F_1, F_2, \ldots, F_{2n-1}$ are values of some nth-order exponential polynomial then by expanding fraction (7) in negative powers of m, one can compute the exact values of this exponential polynomial, $F_{2n}, F_{2n+1}, F_{2n+2}, F_{2n+3}, \ldots$

For this, the equally spaced values of the function under consideration can be computed by means of the recursion formula

$$F_{n+i} + A_1 F_{n-1+i} + A_2 F_{n-2+i} + \cdots + A_n F_i = 0$$

$$(i = 0, 1, 2, 3, \ldots), \quad (8)$$

which is obtained from the equation

$$m^n + A_1 m^{n-1} + A_2 m^{n-2} + \cdots + A_n = 0 \quad (9)$$

by using the following table of correspondences:

$$m^0 \to F_0, \quad m \to F_1,$$

$$m^2 \to F_2, \ldots, m^i \to F_i, \ldots \quad (10)$$

Thus, one can solve the problem of extrapolating an exponential polynomial given 2n equally-spaced values.

This fact allows one to develop a sufficiently simple method of calculating transient responses by means of recursion formulas.

Method of Constructing a Recursion Equation for Equally Spaced Values of a Time-Domain Function

Recursion equation (8), as was mentioned, can be obtained by means of Eq. (9). The coefficients of this equation, $A_1, A_2, \ldots, A_n$, are expressed, as is well known, in terms of its roots $e^{p_k \tau}$:

$$A_1 = -(e^{p_1\tau} + e^{p_2\tau} + \cdots + e^{p_n\tau}),$$
$$A_2 = [e^{(p_1+p_2)\tau} + e^{(p_1+p_3)\tau} + \cdots + e^{(p_{n-1}+p_n)\tau}],$$
$$\cdots\cdots\cdots\cdots$$
$$A_n = (-1)^n e^{(p_1+p_2+\cdots+p_n)\tau}.$$

It follows from this that coefficient A_n of this equation can be exactly determined by coefficient a_1 of Eq. (4):

$$A_n = e^{-a_1\tau}.$$

However, the other coefficient cannot be determined in such fashion. For their computation, therefore, we use an approximate method.

If we express the relationship between parameters p and m in the form of an approximate formula

$$p \approx \frac{2}{\tau}\,\frac{m-1}{m+1},$$

then, after substituting this relationship in Eq. (4), we can obtain an Eq. (9) which is valid for sufficiently small τ.

To increase the size of the discrete step, we transform this equation. The coefficients of Eq. (9) are combinations of the roots $e^{p k \tau}$. By setting up the analogous combinations of the squares of these roots, $e^{2pk\tau}$, we obtain an equation with a double step.

The coefficients of the new, transformed equation are set up by the method, known in algebra, which was employed by N. I. Lobachevskii for solving algebraic equations of high degree:

$$\begin{aligned} A_1^{(1)} &= A_1^2 - 2A_2, \\ A_2^{(1)} &= A_2^2 - 2A_1A_3 + 2A_4, \\ &\cdots\cdots\cdots\cdots \\ A_{n-1}^{(1)} &= A_{n-1}^2 - 2A_{n-2}A_n, \\ A_n^{(1)} &= A_n^2. \end{aligned} \quad (11)$$

We thus obtain the equation for the double step

$$m^{2n} + A_1^{(1)} m^{2n-2} + A_2^{(1)} m^{2n-4} + \cdots + A_n^{(1)} = 0.$$

From this, the recursion formula follows

$$F_{2n+i} + A_1^{(1)} F_{2n-2+i} +$$

$$+ A_2^{(1)} F_{2n-4+i} + \cdots + A_n^{(1)} F_i = 0.$$

By using this method of doubling the step several times, we can increase the discrete step by factor of 4, 8, 16, etc., without decreasing the accuracy of the recursion formula.

With this, the given method can be applied both at the beginning and during the course of the computation of the transient characteristics, thus allowing, on the one hand, a sufficient accuracy of the recursion formula to be obtained and, on the other hand, a significant reduction in the amount of computational work to be realized.

The initial equally spaced values of the time-domain function sought, these being necessary for the use of recursion formulas, may be first of all computed by means of the series

$$F(t) = s_0 + s_1 t + s_2 \frac{t^2}{2!} + s_3 \frac{t^3}{3!} + \cdots,$$

which is obtained by expanding the given function in negative powers of p. With this, in order to simplify the computations, one can use equispaced function values for both positive and negative t, i.e., compute $F(\tau)$ and $F(-\tau)$, $F(2\tau)$, and $F(-2\tau)$, etc.

Example of the Computation of Transient Responses

We illustrate the method of computing transient responses by the example of an electrohydraulic servo system acted on by a unit step function.*

In this case, the Laplace transform of the function has the form:

$$\bar{F}(p) = \frac{0.0035\, p^2 + 0.37\, p + 9.5}{p\,[p^4 + 103\, p^3 + 3065\, p^2 + 149250\, p + 1081500]}$$

By separating the pole of the function for p = 0, we get

$$\bar{F}(p) = \frac{1}{1081500} \left\{ \frac{9.5}{p} - \bar{F}_1(p) \right\},$$

where

$$\bar{F}_1(p) = \frac{9.5\, p^3 + 978.5\, p^2 + 25332.25\, p + 1017720}{p^4 + 103\, p^3 + 3065\, p^2 + 149250\, p + 1081500}.$$

The computation of the curve of the transient response will be carried out for the function $\bar{F}_1(p)$. If, in the system's characteristic equation

$$p^4 + 103\, p^3 + 3065\, p^2 + 149250\, p + 1081500 = 0$$

we make the substitution

$$p = \frac{2}{\tau} \frac{m-1}{m+1},$$

then, for the step $\tau = 0.005$, we shall have

$$m^4 - 3.5263\, m^3 + 4.6615\, m^2 - 2.7283\, m + 0.5937 = 0.$$

By employing the method of doubling the step, we get

$$m^8 - 3.1118\, m^6 + 3.6754\, m^4 - 1.9086\, m^2 + 0.3525 = 0.$$

Thus, the recursion formula for the step $\tau = 0.01$ will have the form:

$$F_{4+i} =$$

$$= 3.1118\, F_{3+i} - 3.6754\, F_{2+i} + 1.9086\, F_{1+i} - 0.3525\, F_i.$$

We determine the initial values of the function sought by means of the series

$$F_1(t) = 9.5 - 1892.5\, t^2 - 1712\, t^3 + 11221\, t^4.$$

We have $F_0 = 9.500$, $F_1 = 9.309$, $F_2 = 8.731$, and $F_3 = 7.760$. Further computations are carried out then by means of the recursion formulas for the step $\tau = 0.01$ and also for $2\tau = 0.02$ and $4\tau = 0.04$:

$$F_{2(4+i)} = 2.3325\, F_{2(3+i)} - 2.3352\, F_{2(2+i)} + 1.0516\, F_{2(1+i)} - 0.1243\, F_{2i},$$

$$F_{4(4+i)} = 0.770\, F_{4(3+i)} - 0.796\, F_{4(2+i)} + 0.525\, F_{4(1+i)} - 0.0155\, F_{4i}.$$

The results of these computations are given in the table.

The values obtained are close to the values computed by other methods (cf. the graph on page 50 of the book by V. V. Solodovnikov et al., Frequency Methods of Constructing Transient Responses using Tables and Nomograms.) This accuracy can be increased if the initial discrete step is decreased.

This example shows the efficacy of the method of doubling the step since, for computing this same curve of the transient response with about the same accuracy using the other approximate methods cited at the beginning of this paper, a significantly greater number of points would have to be computed.

Method of Determining the Denominator of the Transform of an Approximating Function

We now pose the problem of determining the Laplace transform of an exponential polynomial used to approximate a given time-domain characteristic of a system.

The first step in the solution of this problem will be the determination of the denominator of the fraction expressing the transform being sought.

Let the given time-domain characteristic be approximated by an exponential polynomial for 2n equally spaced interpolation points. Then, function (7) can be found for the given equispaced values of the time-domain characteristic. Using this function, we can determine any number of equally spaced values of the exponential nth-order polynomial which assumes the given values at the selected interpolation points.

It is necessary for this that the function found be expanded in negative powers of m, or that recursion formula (8) be employed.

These equally spaced values can be used for the determination of the coefficients of the system's characteristic equation

$$p^n + a_1 p^{n-1} + a_2 p^{n-2} + \cdots + a_n = 0.$$

* This example was borrowed from the book by V. V. Solodovnikov, Yu. I. Topcheev, and G. V. Krutikova, Frequency Methods of Constructing Transient Responses using Tables and Nomograms [in Russian] (Gostekhizdat, 1955).

Table of Values of the Function F(t), Computed from the Formula

$$F(t) = \frac{1}{1081500}\{9.5 - F_1(t)\}$$

t	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.08	0.10
$F(t)\cdot 10^8$	0.000	0.177	0.711	1.608	2.752	4.010	5.232	7.189	8.016
t	0.12	0.16	0.20	0.24	0.28	0.32	0.36	0.40	0.44
$F(t)\cdot 10^6$	7.733	6.236	6.918	8.858	8.998	7.956	8.419	8.978	9.086

If we use approximate formulas for the relationship between the parameters m and p which give convergent fractions for the logarithm, for example,

$$p \approx \frac{2}{\tau}\frac{m-1}{m+1}, \quad p \approx \frac{3}{\tau}\frac{m^2-1}{m^2+4m+1},$$

$$p \approx \frac{4}{\tau}\frac{(m-1)(m^2+4m+1)}{(m+1)(m^2+10m+1)},$$

and substitute them in the sought system's characteristic equation, we then obtain, after some transformations,

$$m^n + A_1^* m^{n-1} + A_2^* m^{n-2} +$$

$$+ \cdots + A_{n-1}^* m + A_n^* = 0,$$

or

$$m^{2n} + D_1 m^{2n-1} + D_2 m^{2n-2} +$$

$$+ \cdots + D_{2n-1} m + D_{2n} = 0,$$

or

$$m^{3n} + G_1 m_{3n-1} + G_2 m^{3n-2} +$$

$$+ \cdots + G_{3n-1} m + G_{3n} = 0.$$

By using these equations and the table of correspondences given in (10), we can set up the recursion relationships connecting a definite number of equispaced values of the exponential polynomial sought. Thus, we can obtain

$$F_{n+i} + A_1^* F_{n-1+i} + A_2^* F_{n-2+i} +$$

$$+ \cdots + A_{n-1}^* F_{1+i} + A_n^* F_i = 0,$$

or

$$F_{2n+i} + D_1 F_{2n-1+i} + D_2 F_{2n-2+i} +$$

$$+ \cdots + D_{2n-1} F_{1+i} + D_{2n} F_i = 0,$$

or

$$F_{3n+i} + G_1 F_{3n-1+i} + G_2 F_{3n-2+i} +$$

$$+ \cdots + F_{3n-1} F_{1+i} + G_{3n} F_i = 0.$$

By means of these recursion formulas, one can determine the unknowns $a_2, a_3, \ldots, a_n$ which enter into the coefficients A_k^* | D_k, G_k |. We note that coefficient a_1 equals

$$a_1 = -\frac{1}{\tau}\ln|A_n|.$$

For example, for the interpolation of a given time-domain characteristic, let there be used a second-order exponential polynomial which is defined by the four equispaced values F_0, F_1, F_2, and F_3. Then, the function $\overline{\Phi}(m)$ will have the form:

$$\overline{\Phi}(m) = \frac{B_0 m + B_1}{m^2 + A_1 m + A_2}.$$

By means of this function, we can determine F_4, F_5, etc. Then, for the determination of the coefficients of the second-order characteristic equation

$$p^2 + a_1 p + a_2 = 0$$

by using the approximate equality

$$p \approx \frac{3}{\tau}\frac{m^2-1}{m^2+4m+1}$$

we obtain the following equation:

$$\frac{9}{\tau^2}(F_{4+i} - 2F_{2+i} + F_i) +$$

$$+ a_1\frac{3}{\tau}(F_{4+i} + 4F_{3+i} - 4F_{1+i} - F_i) +$$

$$+ a_2(F_{4+i} + 8F_{3+i} + 18F_{2+i} + 8F_{1+i} + F_i) = 0,$$

where

$$a_2 = -\frac{1}{\tau}\ln|A_2|.$$

For characteristic equations of higher order, the coefficients are determined from a system of linear equations.

Method of Determining the Numerator of the Transform

The determination of the transform's numerator can be carried out by means of series whose coefficients are the initial values of the approximating function sought and its derivatives, or by the initial moments of this function.

We consider the series obtain by expanding the function $m\overline{\Phi}(m)$ in negative powers of m:

$$m\overline{\Phi}(m) =$$

$$= F_0 + \frac{F_1}{m} + \frac{F_2}{m^2} + \cdots + \frac{F_{2n-1}}{m^{2n-1}} + \frac{F_{2n}}{m^{2n}} + \cdots$$

The coefficients of this series are equispaced values of the exponential polynomial being sought.

If we expand the function $m^{-k} = e^{kp\tau}$ in a series and regroup terms, we get

$$\tau e^{p\tau}\overline{\Phi}(e^{p\tau}) =$$

$$= \left(F_0 + \sum_{i=1}^{\infty} F_i\right)\tau - p\sum_{i=1}^{\infty} i\tau^2 F_i + \frac{p^2}{2!}\sum_{i=1}^{\infty} i^2\tau^3 F_i - \cdots =$$

$$= \tau c_0 - pm_1 + \frac{p^2}{2!}m_2 - \frac{p^3}{3!}m_3 + \cdots$$

Here, m_1, m_2, and m_3 are the initial moments of the function.

It is easily seen that the sums entering into the given expansion are approximate expressions for the initial moments as defined by the trapezoidal quadrature formula.

Thus, we shall have

$$\widehat{F}(p) \approx \tau\left(c_0 - \frac{F_0}{2}\right) - pm_1 + \frac{p^2}{2!}m_2 - \frac{p^3}{3!}m_3 + \cdots$$

An analogous representation can be obtained by first carrying out an expansion of the functions $e^{kp\tau}$ in the numerator and denominator of the fraction for $m\overline{\Phi}(m)$ and the expanding this fraction in positive powers of p, i.e.,

$$\tau e^{p\tau}\overline{\Phi}(e^{p\tau}) = \tau\frac{(B_0 + B_1 + \cdots + B_{n-1}) - p\tau[nB_0 + (n-1)B_1 + \cdots + B_{n-1}] + \cdots}{(1 + A_1 + \cdots + A_n) - p\tau[n + (n-1)A_1 + \cdots + A_{n-1}] + \cdots} =$$

$$= \tau c_0 - pm_1 + \frac{p^2}{2!}m_2 - \frac{p^3}{3!}m_3 + \cdots$$

It should be mentioned that the accuracy of the computations of the initial moments by such a method is higher, the higher the order of the moment.

While this assertion can be rigorously proven, it is better to illustrate it by a simple example.

Let it be required to determine the initial moments of the function e^{-t}. For this purpose, we expand the transform of the function in positive powers of p:

$$\frac{1}{p+1} = 1 - p + p^2 - p^3 + p^4 - \cdots$$

We now consider the interpolation of the given function by two points, $t = 0$ and $t = 1$. The function $\tau m\overline{\Phi}(m)$ will then have the form

$$\frac{m\tau}{m - 0.36788} = \frac{\tau e^{p\tau}}{e^{p\tau} - 0.36788}.$$

If we expand this function in positive powers of p we get

$$\tau e^{p\tau}\overline{\Phi}(e^{p\tau}) = 1.58198 - 0.92068p + 0.99615p^2 -$$

$$- 1.00108p^3 + 1.00012p^4 - \ldots \quad (\tau = 1).$$

By comparing the coefficients of the two expansions we easily see that the accuracy in the determination of the moments rapidly increases with increasing order of the moment, even for a significantly large discrete step.†

To determine the exact values of the lower-order moments, we can find an approximating fraction for the segment of the series whose coefficients are accurately defined, and expand this fraction in decreasing powers of p. By extending the expansion of this fraction, we can find the initial values of the function sought, plus those of its derivatives.

The method given permits the determination, with very high accuracy, of both the numerator and denominator of the first fraction expressing the transform of the approximating function sought.

To determine the fraction which approximates a segment of a series, one can also use the denominator of the fraction obtained by the method of the previous paragraph.

Thus, one can obtain the series

$$\overline{F}(p) = m_0 - m_1 p + m_2\frac{p^2}{2!} - m_3\frac{p^3}{3!} + \cdots$$

† The initial moments are proportional to the corresponding coefficients of the series.

The coefficients of the numerator of the Laplace transform are determined from the coefficients of this series:

$$b_{n-1} = a_n m_0,$$
$$b_{n-2} = -a_n m_1 + a_{n-1} m_0,$$
$$b_{n-3} = a_n \frac{m_2}{2!} - a_{n-1} m_1 + a_{n-2} m_0,$$
$$\cdots\cdots\cdots\cdots\cdots$$

<u>Example.</u> Let it be required to determine the Laplace transform for the exponential polynomial which is characterized by the values

$$F_0 = 0, \quad F_1 = 1, \quad F_2 = 1, \quad F_3 = 0.75 \quad \text{for} \quad \tau = 1.$$

<u>Solution.</u> For the given equispaced values, we construct the series

$$\overline{F}_1(m) = \frac{1}{m^2} + \frac{1}{m^3} + \frac{0.75}{m^4} + \cdots,$$

for which we determine the approximating second-order fraction:

$$\overline{\Phi}(m) = \frac{1}{m^2 - m + 0.25}.$$

By using the method presented earlier, we find the denominator of the transform, in the form

$$Q(p) = p^2 + 1.3863\,p + 0.4793.$$

To determine the numerator of the transform, we use the expansion

$$\frac{e^p}{0.25 - e^p + e^{2p}} =$$

$$= 4 - 12\,p + 26\,p^2 - 50\,p^3 + 90.167\,p^4 - 156.1\,p^5 + \cdots,$$

by means of which we find the zeroth-order initial moment: $m_0 = 4.18$.

The final solution of the problem is the expression

$$\overline{F}(p) = \frac{2.0034}{p^2 + 1.3863\,p + 0.4793}.$$

The exact transform of the initial function has the form:

$$\overline{F}(p) = \frac{2}{p^2 + 1.3863p + 0.4804} \to 2te^{-0.6931t}.$$

In conclusion, we note that this same problem could be solved by computing moments of sufficiently high order.

In particular, by using the moments of second, third, fourth, and fifth order, we get

$$\overline{F}(p) = \frac{2.0025}{p^2 + 1.3836\,p + 0.47907}.$$

CONCLUSION

1. The possibility was demonstrated of calculating transient responses by means of recursion formulas obtained on the basis of an approximate formula for the differentiation operator and the application of the method of doubling the discrete step.

The method of calculating transient responses presented here is highly accurate.

2. A method was presented of determining the Laplace transform corresponding to an exponential polynomial by interpolating the given time-domain characteristic for equally spaced points. This method can be used for finding the Laplace transforms of functions which are given graphically.

LITERATURE CITED

[1] G. S. Pospelov, Fundamentals of Automation [in Russian] (Izd. Zhukovskii VVIA, 1954).

[2] R. Boxer and S. Thaler, "A simplified method of solving linear and nonlinear systems," Proc. I.R.E. No. 1, (1956).

[3] R. Boxer, "A note on numerical transform calculus, Proc. I.R.E. No. 10 (1957).

[4] A. A. Markov, "Lectures on continued fractions," in: Collected Works on the Theory of Continued Fractions and the Theory of Functions Deviating Least from Zero [in Russian] (Gostekhizdat, 1948).

A GENERATOR OF RANDOM PROCESSES FROM THEIR GIVEN SPECTRAL DENSITY MATRICES

I. Matyash and Ya. Shilkhanek

(Pardubitse, Czechoslovakia)

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A method is described for constructing a generator of n stationary random processes with arbitrary matrices of rational spectral densities, using a minimal number of uncorrelated white noise generators and stable linear filters. The transfer functions of these filters are determined by a very simple method from the given spectral densities. The method is illustrated by an example for the case when n = 3.

Analog computers are frequently used for the analysis and synthesis of automatic control and regulation systems which are acted upon by random functions of time. For this, the inputs to the computer must be random functions which correspond to the input random functions of the actual system.

In this paper we shall consider only stationary random functions (processes) whose proper and mutual spectral densities are rational-fractional functions of the variable $s = j\omega$. In practice it is necessary to replace the experimentally determined spectral densities by approximating rational functions. Methods of implementing such approximations are described in the literature (cf., for example, [1, 2]).

To study actual systems by means of analog computers (modeling devices), it is necessary to generate random processes with given spectral densities. The generation of one stationary random process with a given proper spectral density is described, for example, in [2]. In [3] a method is given for reproducing two processes with given proper and mutual spectral densities. However, with this method one can obtain two random processes with given spectral densities only in special cases

In the present paper we present a general method for generating any number n of stationary random processes with arbitrary proper and mutual spectral densities.

Generation of One Random Process

The generator's block schematic (cf. [2]) is shown in Fig. 1, where Q_1 is a white noise generator (a process with unit spectral density) and F_{11} is a filter with transfer function $Y_{11}(s)$.

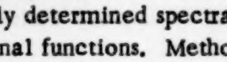

Fig. 1.

The condition that the output signal $u_1(t)$ be a process with the given density ${}^1G_{11}(s)$ is written in the form

$$ {}^1G_{11}(s) = Y_{11}(s)\,Y_{11}(-s). \qquad (1)$$

Since ${}^1G_{11}(s)$ is a spectral density, there exists just one function $A_1(s)$ which does not have poles and zeros in the right half-plane and such that the following equation holds

$$A_1(s)\,A_1(-s) = {}^1G_{11}(s) \qquad (2)$$

(cf. [1, 2, 4]). By comparing this with (1), we obtain the well-known result (cf. [2])

$$Y_{11}(s) = A_1(s), \qquad (3)$$

wherein $Y_{11}(s)$ is the transfer function of a stable linear filter with minimal phase.

For the generation of several processes, it is sometimes necessary to construct a filter with nonminimal phase. If we connect in series with the filter whose transfer function is $A_1(s)$ a filter which varies only in phase, and whose transfer function has the form:

$$H_1(s) = \frac{f(s)}{f(-s)}, \qquad (4)$$

where f(s) is a polynomial then, obviously, the following relationship holds

$$|H_1(s)|^2 = H_1(s)\,H_1(-s) = 1, \qquad (5)$$

i.e., the spectral density of the output process is not changed.

Thus, the transfer function $Y_{11}(s)$ of filter F_{11} can be written in the general form:

$$Y_{11}(s) = A_1(s)\,H_1(s), \qquad (6)$$

where $H_1(s)$ is any function of the form of (4) such that the function $A_1(s)H_1(s)$ is the transfer function of a stable filter.

Generation of n Random Processes

Spectral densities of the random processes. We shall denote by ${}^1G_{ik}(s)$ (i, k = 1, 2, . . . , n) the given proper and mutual spectral densities of the random processes $u_1(t)$, $u_2(t)$, . . . , $u_n(t)$. The functions ${}^1G_{ik}(s)$ denote the mutual spectral density of the processes $u_i(t)$ and $u_k(t)$. The functions ${}^1G_{ik}(s)$ form the matrix of spectral densities $\| {}^1G_{ik}(s) \|$ (i, k = 1, 2, . . . , n) [5].

From the self-evident relationship

$$^1G_{ki}(s) = {}^1G_{ik}(-s) \qquad (s = j\omega) \qquad (7)$$

we obtain

$$^1G_{ki}(s) = {}^1G^*_{ik}(s)\,, \qquad (8)$$

where the symbol * denotes the complex conjugate quantity. Consequently, the spectral density matrix $\|{}^1G_{ik}(s)\|$ is a Hermitian matrix. It is completely defined by its $n(n+1)/2$ elements $^1G_{ik}(s)$ $(i = 1, 2, \ldots, n;\ k = 1, i+1, \ldots, n)$.

The Appendix describes several properties of the spectral density matrix which are used in this paper.

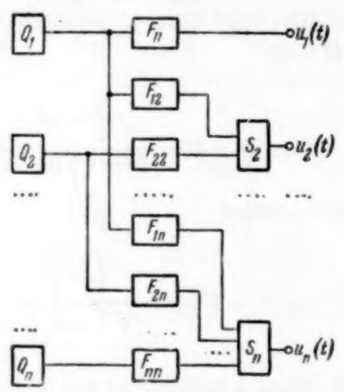

Fig. 2.

Block schematic of the generator. The block schematic of the generator of n random processes is shown in Fig. 2. It consists of n uncorrelated (for example, statistically independent) white noise generators $Q_1, Q_2, \ldots, Q_n$ (generators of uncorrelated processes with unit spectral densities), $n(n+1)/2$ stable linear filters F_{ik} $(i = 1, 2, \ldots, n;\ k = i, i+1, \ldots, n)$ with transfer functions $Y_{ik}(s)$ and $n-1$ adders S_k $(k = 2, 3, \ldots, n)$. Process $u_1(t)$ is found at the output of filter F_{11}, while process $u_k(t)$ $(k = 2, 3, \ldots, n)$ is found at the output of adder S_k.

In order that the generator's output signals $u_1(t), u_2(t), \ldots, u_n(t)$ be the processes with the given spectral density matrix $\|{}^1G_{ik}(s)\|$ $(i, k = 1, 2, \ldots, n)$, it is necessary (cf., for example, [6]) that the following conditions hold:

$$\begin{aligned}
{}^1G_{11}(s) &= Y_{11}(s)\,Y_{11}(-s),\\
{}^1G_{12}(s) &= Y_{12}(s)\,Y_{11}(-s),\\
{}^1G_{22}(s) &= Y_{12}(s)\,Y_{12}(-s) + Y_{22}(s)\,Y_{22}(-s),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

i.e., in general,

$$^1G_{ik}(s) = \sum_{j=1}^{i} Y_{jk}(s)\,Y_{ji}(-s)$$

$$\begin{pmatrix} i = 1, 2, \ldots, n \\ k = i, i+1, \ldots, n \end{pmatrix}. \qquad (9)$$

The problem thus reduces to determining the transfer functions $Y_{ik}(s)$ of the stable linear filters F_{ik} $(i = 1, 2, \ldots, n;\ k = i, i+1, \ldots, n)$ such that system of equation (9) holds.

Determination of the transfer functions $Y_{ik}(s)$. From (9), by setting $i = 1$, we obtain

$$^1G_{1k}(s) = Y_{1k}(s)\,Y_{11}(-s) \quad (k = 1, 2, \ldots, n). \qquad (10)$$

For $k = 1$, Eq. (10) is identical with (1). Consequently, in accordance with formula (6),

$$Y_{11}(s) = A_1(s)\,H_1(s). \qquad (11)$$

By using (10), we find the functions $Y_{1k}(s)$ in the form

$$Y_{1k}(s) = \frac{^1G_{1k}(s)}{A_1(-s)}\,H_1(s) \qquad (k = 1, 2, \ldots, n). \qquad (12)$$

The function $H_1(s)$, of the form given in (4), is such that all the $Y_{1k}(s)$ $(k = 1, 2, \ldots, n)$ are transfer functions of stable filters. We obtain a function $H_1(s)$ of the least degree if, for the null points of $H_1(s)$ [the roots of the polynomial $f(s)$], we choose the set of poles in the right half-plane of the expressions $^1G_{1k}(s)/A_1(-s)$ (each pole being chosen with the maximum multiplicity in these expressions). This completely determines the function $H_1(s)$ [cf. (4)].

To compute the transfer functions $Y_{ik}(s)$ $(i > 1)$, we define the auxiliary functions $^{r+1}G_{ik}(s)$ $(r = 1, 2, \ldots, n-1)$ by the formula

$$^{r+1}G_{ik}(s) = {}^1G_{ik}(s) - \sum_{j=1}^{r} Y_{jk}(s)\,Y_{ji}(-s)$$

$$\begin{pmatrix} i = r+1, \ldots, n \\ k = i, \ldots, n \end{pmatrix}. \qquad (13)$$

These functions obviously satisfy the recursion relationship

$$^{r+1}G_{ik}(s) = {}^rG_{ik}(s) - Y_{rk}(s)\,Y_{ri}(-s). \qquad (14)$$

After substituting (13) in (9) we obtain, for $r+1 = i$,

$$^iG_{ik}(s) = Y_{ik}(s)\,Y_{ii}(-s)$$

$$(k = i,\ i+1, \ldots, n). \qquad (15)$$

For $k = i$ we obtain from this

$$^iG_{ii}(s) = Y_{ii}(s)\,Y_{ii}(-s). \qquad (16)$$

If follows from (16) that the function $^iG_{ii}(s)$ $(i = 2, 3, \ldots, n)$ must be real and nonnegative:

$$^iG_{ii}(s) \geqslant 0 \qquad \begin{pmatrix} s = j\omega \\ i = 2, 3, \ldots, n \end{pmatrix}. \qquad (17)$$

It is proven in the Appendix that conditions (17) are met if the functions $^1G_{ik}(s)$ form a spectral density matrix.

Since formula (16) has the form of (1), we define the functions $Y_{ii}(s)$ in accordance with formula (6) by the method described earlier:

$$Y_{ii}(s) = A_i(s) H_i(s), \tag{18}$$

where the functions $A_i(s)$ are defined by the equations

$$A_i(s) A_i(-s) = {}^iG_{ii}(s) \tag{19}$$

just as $A_1(s)$ of (2), and the $H_i(s)$ are as yet undetermined functions of the form of (4).

By substituting (18) in (15), we get

$$Y_{ik}(s) = \frac{{}^iG_{ik}(s)}{A_i(-s)} H_i(s)$$

$$\begin{pmatrix} i = 1, 2, \ldots, n \\ k = i, i+1, \ldots, n \end{pmatrix}. \tag{20}$$

The functions $H_i(s)$ are defined by the conditions that all the $Y_{ik}(s)$ ($k = i, i+1, \ldots, n$) be the transfer functions of stable filters [analogously to the function $H_1(s)$].

Formulas (20) define all the transfer functions $Y_{ik}(s)$ ($i = 1, 2, \ldots, n$; $k = i, i+1, \ldots, n$). In practice, one first computes transfer functions $Y_{1k}(s)$ from the given functions ${}^1G_{1k}(s)$ ($k = 1, 2, \ldots, n$), then determines the auxiliary functions ${}^2G_{ik}(s)$ from (14), then computes from these the transfer functions $Y_{2k}(s)$ ($k = 2, 3, \ldots, n$), etc. In general, the functions ${}^{r+1}G_{ik}(s)$ are determined step by step from the ${}^rG_{ik}(s)$ by formula (14). Transfer functions $Y_{r+1,k}(s)$ are computed from these by formulas (20).

<u>Example.</u> As an illustration, we use the example of the generator of three random processes whose spectral densities are given in the form

$${}^1G_{11}(s) = \frac{2-s^2}{4-s^2}, \qquad {}^1G_{12}(s) = \frac{s^2}{(3+s)(2-s)},$$

$${}^1G_{13}(s) = \frac{2-s^2}{(3+s)(2-s)}, \qquad {}^1G_{22}(s) = \frac{-2s^2+2s^4}{(2-s^2)(9-s^2)},$$

$$G_{23}(s) = \frac{-2s+2s^2}{9-s^2}, \qquad {}^1G_{33}(s) = \frac{4-2s^2}{3-s^2}.$$

It follows from (12) that

$$Y_{11}(s) = \frac{\sqrt{2}+s}{2+s} H_1(s),$$

$$Y_{12}(s) = \frac{s^2}{(3+s)(\sqrt{2}-s)} H_1(s),$$

$$Y_{13}(s) = \frac{\sqrt{2}+s}{3+s} H_1(s).$$

From the stability conditions we obtain, based on what has been presented earlier,

$$H_1(s) = \frac{\sqrt{2}-s}{\sqrt{2}+s}.$$

By this, transfer functions $Y_{1k}(s)$ ($k = 1, 2, 3$) are completely determined.

From formula (14) we now compute the functions ${}^2G_{ik}(s)$:

$${}^2G_{22}(s) = \frac{-s^2}{9-s^2}, \qquad {}^2G_{23}(s) = \frac{-2s+s^2}{9-s^2},$$

$${}^2G_{33}(s) = \frac{(2-s^2)(15-s^2)}{(3-s^2)(9-s^2)}.$$

From these, by (20), we obtain

$$Y_{22}(s) = \frac{s}{3+s} H_2(s), \qquad Y_{23}(s) = \frac{2-s}{3+s} H_2(s).$$

Consequently, we can set $H_2(s) = 1$. This then determines functions $Y_{22}(s)$ and $Y_{23}(s)$.

Finally, the following expression for ${}^3G_{33}(s)$ follows from (14)

$${}^3G_{33}(s) = 10 \frac{\frac{9}{5} - s^2}{(3-s^2)(9-s^2)}.$$

From whence

$$Y_{33}(s) = \sqrt{10}\, \frac{\sqrt{\frac{9}{5}} + s}{(\sqrt{3}+s)(3+s)}.$$

The block schematic of the generator of the three random processes desired is shown in Fig. 3.

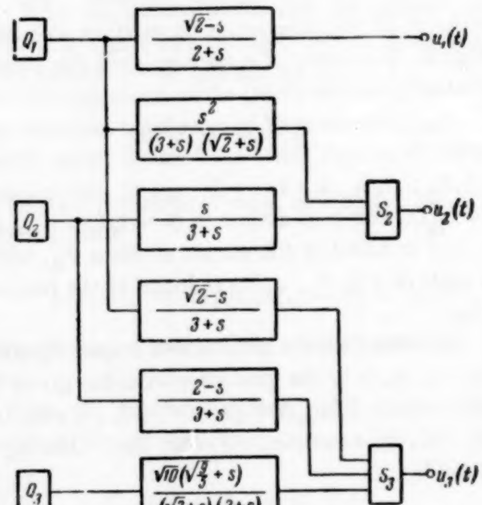

Fig. 3.

CONCLUSIONS

The method herein described for generating any number $\underline{n}$ of stationary random processes from a given matrix of rational spectral densities is a general method. It permits one to construct a generator of random processes from any matrix of rational spectral densities.

One easily convinces oneself that the generator contains the least number (n) of white noise generators Q_i and the least number ($n(n+1)/2$) of linear filters F_{ik}.

(In [3] there was described a generator of two processes which consisted of three white noise generators and three filters.)

The generator's functional schematic (Fig. 2) allows the transfer functions $Y_{ik}(s)$ of the filters F_{ik} to be computed in a very simple way from the given spectral densities [cf. formulas (20)].

APPENDIX

Certain Properties of Spectral Density Matrices

For brevity, we shall formulate these properties in the form of theorems. From these properties one can derive results which are important for the construction of the random process generators described in this paper.

Theorem 1. The functions ${}^{r+1}G_{ik}(s)$ ($i = r + 1, \ldots, n$; $k = 1, \ldots, n$) of Eq. (13) are defined by the relationships

$$ {}^{r+1}G_{ik} = \frac{\begin{vmatrix} {}^1G_{11} \ldots {}^1G_{1r}\, {}^1G_{1k} \\ \ldots \quad \ldots \quad \ldots \\ {}^1G_{r1} \ldots {}^1G_{rr}\, {}^1G_{rk} \\ {}^1G_{i1} \ldots {}^1G_{ir}\, {}^1G_{ik} \end{vmatrix}}{\Delta_r}, \tag{21} $$

where

$$ \Delta_r = \begin{vmatrix} {}^1G_{11} \ldots {}^1G_{1r} \\ \ldots \quad \ldots \quad \ldots \\ {}^1G_{1r} \ldots {}^1G_{rr} \end{vmatrix}. \tag{22} $$

Proof. From (14), we get that

$$ {}^{r+1}G_{ik} = \frac{\begin{vmatrix} {}^rG_{rr}\; {}^rG_{rk} \\ {}^rG_{ir}\; {}^rG_{ik} \end{vmatrix}}{{}^rG_{rr}}. \tag{23} $$

The assertion of our theorem is proven by the method of complete induction. It follows from (23) that (21) is valid for $r = 1, 2, \ldots$ We assume that (21) is valid for all $r \leq m-1$. For $r = m$, we obtain from (23) that

$$ {}^{m+1}G_{ik} = \frac{\begin{vmatrix} {}^mG_{mm}\; {}^mG_{mk} \\ {}^mG_{im}\; {}^mG_{ik} \end{vmatrix}}{{}^mG_{mm}}. \tag{24} $$

By assumption, we can assert the relationship

$$ {}^mG_{mm} = \frac{\Delta_m}{\Delta_{m-1}} \tag{25} $$

from whence

$$ \frac{\Delta_m}{\Delta_{m-1}}\, {}^{m+1}G_{ik} = \begin{vmatrix} {}^mG_{mm}\; {}^mG_{mk} \\ {}^mG_{im}\; {}^mG_{ik} \end{vmatrix}. \tag{26} $$

From Sylvester's determinant identity we get

$$ \begin{vmatrix} {}^mG_{mm}\; {}^mG_{mk} \\ {}^mG_{im}\; {}^mG_{ik} \end{vmatrix} = \frac{1}{\Delta_{m-1}} \cdot \begin{vmatrix} {}^1G_{11} \ldots {}^1G_{1m}\; {}^1G_{1k} \\ \ldots \quad \ldots \quad \ldots \\ {}^1G_{m1} \ldots {}^1G_{mm}\; {}^1G_{mk} \\ {}^1G_{i1} \ldots {}^1G_{im}\; {}^1G_{ik} \end{vmatrix}. \tag{27} $$

By comparing (26) and (27) we immediately obtain the equation

$$ {}^{m+1}G_{ik}\, \Delta_m = \begin{vmatrix} {}^1G_{11} \ldots {}^1G_{1m}\; {}^1G_{1k} \\ \ldots \quad \ldots \quad \ldots \\ {}^1G_{m1} \ldots {}^1G_{mm}\; {}^1G_{mk} \\ {}^1G_{i1} \ldots {}^1G_{im}\; {}^1G_{ik} \end{vmatrix} \tag{28} $$

which proves the theorem.

Corollary of Theorem 1. By virtue of (21), we have the equation

$$ {}^rG_{rr} = \frac{\Delta_r}{\Delta_{r-1}} \qquad (r = 2, 3, \ldots, n). \tag{29} $$

Consequently, conditions (17) can be written in the form

$$ \frac{\Delta_r}{\Delta_{r-1}} \geqslant 0 \qquad (r = 2, 3, \ldots, n). \tag{30} $$

Theorem 2. A necessary and sufficient condition for the matrix $\| {}^1G_{ik} \|$ ($i, k = 1, \ldots, n$) to be a spectral density matrix is that it be a nonnegative Hermitian matrix.

For the proof, cf. [5] and [7].

Theorem 3. A necessary and sufficient condition that a Hermitian matrix be nonnegative is that all its principal minors be nonnegative.

For the proof, cf., for example, [8].

Theorem 4. If $\| {}^1G_{ik} \|$ ($i, k = 1, 2, \ldots, n$) is a spectral density matrix then, for any $r = 2, 3, \ldots, n$, the matrix $\| {}^rG_{ik} \|$ ($i, k = r, r+1, \ldots, n$) [cf. (21)] is also a spectral density matrix.

Proof. The validity of the following inequality is a consequence of theorem 3:

$$ \Delta_r \geqslant 0 \qquad (r = 1, 2, \ldots, n). \tag{31} $$

From Sylvester's determinant identity, one easily proves that all the principal minors of matrix $\| {}^rG_{ik} \|$ are nonnegative. We then obtain the assertion of theorem 4 by using theorems 2 and 3.*

Corollary of Theorem 4. If all the matrices $\| {}^rG_{ik} \|$ ($r = 2, 3, \ldots, n$) are spectral density matrices, then inequalities (17) and (30) hold. It suffices for this,

* The validity of theorem 4 follows directly from (13), since it is easy to determine the random processes for which the spectral density matrix is the matrix $\| {}^rG_{ik} \|$ ($i, k = r, \ldots, n$).

by theorem 4, that matrix $\| {}^1G_{ik} \|$ be a spectral density matrix. It follows from this that, using the method described in this paper, one can construct a generator of random stationary processes with any arbitrary matrix of rational spectral densities.

LITERATURE CITED

[1] V. V. Solodovnikov, Introduction to the Statistical Dynamics of Automatic Control Systems [in Russian] (Gostekhizdat, 1952).

[2] J. H. Laning and R. H. Battin, Random Processes in Automatic Control (McGraw-Hill, N. Y., 1956).

[3] O. J. M. Smith, Feedback Control Systems (McGraw-Hill, N. Y., 1958).

[4] N. Wiener, Extrapolation, Interpolation, and Smoothing of Stationary Time Series (John Wiley & Sons, N. Y., 1949).

[5] A. M. Yaglom, "Introduction to the theory of stationary random functions," Uspekhi Mat. Nauk 7, 5 (1952).

[6] V. S. Pugachev, Theory of Random Functions and Their Application to Problems of Automatic Control [in Russian] (Gostekhizdat, 1957).

[7] H. Cramer, "On the theory of stationary random processes," Ann. Math. 41, 1 (1949).

[8] F. R. Gantmakher, Matrix Theory [in Russian] (Gostekhizdat, 1954).

[9] J. Silhanek and J. Matyas, "Generator skupiny n nahodnych procesu s predepsanymi vlastnimi a vzajemnymi spektralnimi hustotami," Ceskoslovenska patentni prihlaska cis. PV 3962-59.

ON THE STABILITY OF SERVO-CONTROLLED SYSTEMS WITH RANDOM STIMULI

P. S. Landa

(Moscow)

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Excitation conditions are found for servo systems containing nonlinear backlash-type elements. The probability of excitation of such a system when noise is present is computed as a function of time.

Due to feedback paths, both designed and parasitic, and also to the various types of nonlinear elements, automatic control systems which are stable for small oscillations can be excited by large disturbances. Such systems with rigid excitation have, for even relatively small fluctuating stimuli, a finite probability of being excited to autooscillation during a given time interval. Therefore, the solution of the problem of computing this probability is of significant interest for the rational choice of the parameters of such systems.

The problem posed may be solved comparatively simply in the case when the object of control is an oscillatory system with one degree of freedom, and the controlling organ contains some nonlinear element, for example, backlash, dry friction, etc.

In this paper we use, as an example, the system whose block schematic is shown in Fig. 1. The dotted line denotes the additional feedback path between the controlled object's motion and the control organ in this system. The character of this feedback is defined by the coefficient β which, in general, can be complex. For simplicity we shall consider β to be real, but the method is easily generalized to the case when β takes on complex values.

We shall assume that the controlling stimulus g(t) varies very slowly in comparison with the system's natural frequency of oscillation. It may then be neglected in writing the system's equations. With this, the system under consideration, as is clear from Fig. 1, is described by the following equations:

$$\ddot{x} + 2\delta\dot{x} + \omega^2 x = kz_1 - \omega^2\zeta(t),$$
$$\dot{z} = b\varepsilon, \quad \varepsilon = \beta x - z, \; z_1 = f(z), \tag{1}$$

where $\zeta(t) = -\omega^2\zeta_1(t)/k$ is a stationary random function with zero mean and f(z) is the backlash characteristic (Fig. 2).

We also assume that

$$b \gg \delta, \tag{2}$$

i.e., the servo motor's time constant is small in comparison with the natural time of the controlled object.

If a = 0 (no backlash), the system becomes linear and the following conditions for oscillatory instability can easily be written for it:

$$\beta < 0, \quad k > \frac{2\delta\omega(1+\alpha^2)}{|\beta|\alpha} = k_0, \tag{3}$$

where $\alpha = \omega/b$ (with this, the excitation is soft).

If $a \neq 0$, the system becomes stable for small oscillations about the equilibrium position. However, for definite values of k, even if conditions (3) do not hold, rigid excitation of oscillations is possible in the system.

In this case, we shall find the excitation conditions and the system's steady-state oscillations in the absence of noise, and also the probability of system excitation as a function of time and of the magnitude of the noise.

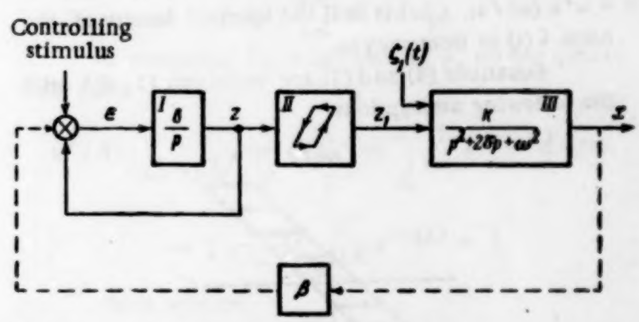

Fig. 1. Block schematic of a servo system. I is a servo motor of the hydraulic amplifier type, II is a nonlinear element of the backlash type, and III is the controlled object.

For this, we use the method presented in [1, 2], and we write the contracted equations for the amplitude and phase of the oscillations, wherein we define the latter in the following way:

$$x = A\cos(\omega t + \varphi), \quad \dot{x} = -A\omega\sin(\omega t + \varphi). \tag{4}$$

By virtue of assumption (2), we can express the coordinate z in terms of A and φ:

$$z = \frac{\beta}{\sqrt{1+\alpha^2}} A\cos(\omega t + \varphi + \psi), \tag{5}$$

where

$$\psi = \text{arc } \operatorname{tg}(-\alpha).$$

By substituting (4) in (5) and expanding the function f(z) = F(t) in a Fourier series in t, we obtain equations in the coordinates (variables) A and φ:

$$\dot{A} = -\delta A - \frac{k\beta C}{2\omega\sqrt{1+\alpha^2}} A \sin\chi + \xi, \quad (6)$$

$$\dot{\varphi} = -\frac{k\beta C}{2\omega\sqrt{1+\alpha^2}} \cos\chi + \frac{\xi'}{A}. \quad (7)$$

Here, we have introduced the following notation:

$$C = \begin{cases} 0 & \text{for } \dfrac{\beta A}{\sqrt{1+\alpha^2}} \leqslant a, \\ \dfrac{1}{\pi}\sqrt{(u^2-1)^2+v^2} & \text{for } \dfrac{\beta A}{\sqrt{1+\alpha^2}} \geqslant a, \end{cases}$$

$$\operatorname{tg}\chi = \alpha\pi\frac{F_\alpha(u)}{v+\alpha(u^2-1)}, \quad u = 1 - 2\frac{a\sqrt{1+\alpha^2}}{|\beta| A},$$

$$v = \frac{3\pi}{2} - \arcsin u + u\sqrt{1-u^2},$$

$$F_\alpha(u) = \begin{cases} 0 & \text{for } \dfrac{\beta A}{\sqrt{1+\alpha^2}} \leqslant a, \\ \dfrac{1}{\alpha\pi}(\alpha v + 1 - u^2) & \text{for } \dfrac{\beta A}{\sqrt{1+\alpha^2}} \geqslant a, \end{cases}$$

ξ and ξ' are δ-correlated random functions with zero means and correlation function $K(\tau) = 2\lambda^{-1}\delta(\tau)$, where $\lambda^{-1} = \omega^2\kappa(\omega)/4$; $\kappa(\omega)$ is half the spectral density of the noise $\zeta(t)$ at frequency ω.

Equations (6) and (7) are valid (cf. [1, 2]) with the following assumptions.

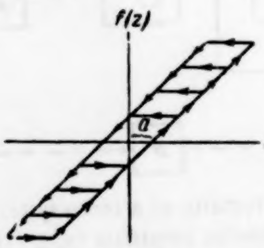

Fig. 2.

1. The oscillations in the system are almost harmonic, i.e., A and φ slowly vary during the period. This requirement corresponds to the condtions:

$$\frac{\delta}{\omega} < 1; \qquad \frac{k|\beta|}{\omega^2(1+\alpha^2)} < 1,$$

$$\begin{cases} \sqrt{\overline{\zeta^2}} < A_0 & \text{for } \tau_{cor} \geqslant \dfrac{2\pi}{\omega}, \\ \sqrt{\omega\kappa(\omega)} < A_0 & \text{for } \tau_{cor} < \dfrac{2\pi}{\omega}, \end{cases} \quad (8)$$

where A_0 is the amplitude of the unstable limiting cycle and τ_{cor} is the correlation time of the noise $\zeta(t)$.

2. The correlation time of the noise is small in comparison with the system's relaxation time, i.e.,

$$\tau_{cor} < \frac{1}{\delta} \text{ and } \tau_{cor} < \frac{\omega(1+\alpha^2)}{k|\beta|}.$$

Since the right side of Eq. (7) does not depend on φ, this expression can be considered as the correction to the system's frequency of oscillations:

$$\Delta\omega = -\frac{k\beta C}{2\omega\sqrt{1+\alpha^2}}\cos\chi + \frac{\xi'}{A}.$$

The presence of the random stimulus in the right side leads only to fluctuations of the frequency about the mean value

$$\omega_0 = \omega + \Delta\omega_{me},$$

where

$$\Delta\omega_{me} = -\frac{k\beta C}{2\omega\sqrt{1+\alpha^2}}\cos\chi.$$

By virtue of conditions (8), $\Delta\omega_{me} \ll \omega$.

To answer the question of system stability, it suffices to investigate only Eq. (6), since it does not depend on φ.

With no noise present, the magnitudes of the limiting cycles are determined from the equations

$$\frac{k\beta}{2\omega\sqrt{1+\alpha^2}} C\sin\chi = -\delta$$

or

$$F_\alpha(u) = -\frac{2\omega\delta(1+\alpha^2)}{k\beta\alpha}. \quad (9)$$

The graph of the function $F_\alpha(u)$ as a function of u for various values of α is given on Fig. 3. The function $F_\alpha(u)$ has an extremum at the point $u_0 = \alpha/\sqrt{(1+\alpha^2)}$, equal to $F_\alpha(u_0) > 1$. Consequently, as is clear from the graph, limiting cycles are possible with the conditions

$$\beta < 0, \quad k > \frac{2\omega\delta(1+\alpha^2)}{|\beta|\alpha F_\alpha(u_0)} = \frac{k_0}{F_\alpha(u_0)}.$$

The points of intersection of the line defined by the right member of Eq. (9) with the curve $F_\alpha(u)$ give the amplitudes of the unstable and stable limiting cycles, where the left point corresponds, as is easily shown, to the unstable cycle.

If $k > k_0$, which corresponds to the condition for excitation of oscillations in the absence of noise, then there is no steady-state stable mode for the system. If, however, $k_0/F_\alpha(u_0) < k < k_0$ then, with backlash in the system, it is possible to have autooscillations with stationary amplitude whose excitation occurs rigidly. Using the method suggested by R. L. Stratonovich (cf. [1, 2]) for the approximate solution of the Fokker-Planck equation, we compute the probability of system excitation when it is stimulated by a random force.

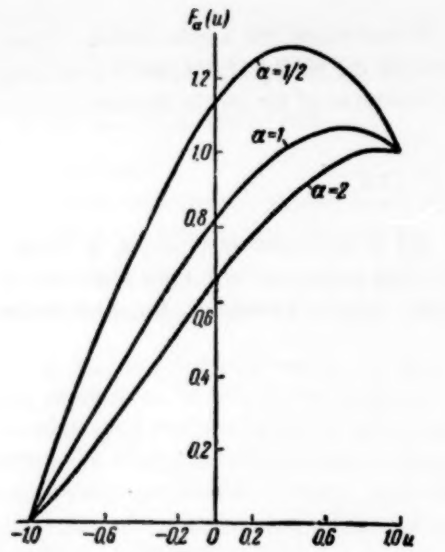

Fig. 3.

It is necessary for this to impose additional conditions on the magnitude of the random stimulus:

$$\frac{1}{\sqrt{\lambda\delta}} < A_0, \quad \frac{1}{\sqrt{\lambda K}} < A_0, \tag{10}$$

where

$$K = -\left.\frac{\partial^2 U(A)}{\partial A^2}\right|_{A=A_0},$$

$$U(A) = \delta\frac{A^2}{2} - \frac{2k\alpha a^2}{\omega|\beta|}\int\frac{F_\alpha(u)}{(1-u)^3}\,du.$$

We now write the corresponding Fokker-Planck equation for the amplitude's distribution density w(A,t):

$$\frac{\partial w}{\partial t} = -\frac{\partial G}{\partial A}, \tag{11}$$

where

$$G = -\left[\delta + \frac{k\beta\alpha F_\alpha(u)}{2\omega(1+\alpha^2)}\right]Aw + \frac{1}{\lambda A}w - \frac{1}{\lambda}\frac{\partial w}{\partial A}.$$

As follows from [2], the probability of system excitation at time $\tau \le t$ equals $q(t) = 1 - e^{-\gamma/2t}$, where

$$\gamma = G_0|_{A=A_0},$$

$$G_0 = -\left[\delta + \frac{k\beta_\alpha F_\alpha(u)}{2\omega(1+\alpha^2)}\right]Aw_0 + \frac{1}{\lambda A}w_0 - \frac{1}{\lambda}\frac{dw_0}{dA},$$

and, as $w_0(A)$, one can take as an approximation the steady-state solution of the Fokker-Planck equation with zero boundary conditions:

$$\frac{dG_0}{dA} = 0,$$

$$w_0(A_0) = 0, \quad \int_0^{A^*} w_0(A)\,dA = 1. \tag{12}$$

By virtue of the zero boundary condition, we obtain the simple expression for γ:

$$\gamma = -\frac{1}{\lambda}\left.\frac{dw_0}{dA}\right|_{A=A_0}.$$

Since, due to conditions (10), we consider the noise to be quite small, we can assume that the effect of the zero boundary condition on the course of the distribution density $w_0(A)$ is limited to a narrow region close to the boundaries characterized by the inequality $\Delta A < \sqrt{1/\lambda K}$. It is natural to assume that, outside of this region, $w_0(A)$ coincides with the steady-state solution of Eqs. (12), w(A), which satisfies the condition that it vanishes at infinity.

As one easily convinces oneself by direct substitution in (12),

$$w(A) = CAe^{-\lambda U(A)}.$$

The normalizing constant C can be computed approximately by using conditions (10):

$$C^{-1} = \int_0^{A_0} Ae^{-\lambda U(A)}\,dA \approx \frac{1}{\lambda\delta}.$$

Based on what has been said earlier, we can set

$$w_0(A) = w(A)\,w^*(A), \tag{13}$$

where $w^*(A)$ is a correction factor, equal to zero for $A = A_0$ and equal to unity for $A_0 - A > 1/\sqrt{\lambda K}$.

By substituting (13) in (12), we obtain the equation for $w^*(A)$:

$$\frac{d}{dA}\left(w\frac{dw^*}{dA}\right) = 0 \tag{14}$$

with the conditions

$$w^*(A_0) = 0, \quad w^*(A) = 1 \text{ for } A_0 - A > \frac{1}{\sqrt{\lambda K}}. \tag{15}$$

We also note that

$$\left.\frac{dw_0}{dA}\right|_{A=A_0} = w(A_0)\left.\frac{dw^*}{dA}\right|_{A=A_0} = -\lambda\gamma.$$

We can now write the first integral of Eq. (14):

$$w(A)\frac{dw^*}{dA} = -\lambda\gamma.$$

To determine the magnitude of γ, we use conditions (15):

$$w^*(A)\Big|_{A_0-A>\sqrt{\frac{1}{\lambda K}}} = C^{-1}\lambda\gamma\int_{A_0}^{A}\frac{1}{A}e^{\lambda U(A)}\,dA \approx$$

$$\approx \gamma\frac{1}{\delta A_0}\sqrt{\frac{\pi}{2\lambda K}}\,e^{\lambda U(A_0)} = 1.$$

From whence

$$\gamma = \delta A_0\sqrt{\frac{2\lambda K}{\pi}}\,e^{-\lambda U(A_0)},$$

where

$$K = \frac{a\alpha k}{A_0\omega\sqrt{1+\alpha^2}}\left.\frac{dF_\alpha(u)}{du}\right|_{A=A_0}.$$

Thus, the probability of system excitation during any given interval of time is completely determined. Knowing it, we can easily compute the average time during which the system is excited. It equals $t_{av} = 2/\gamma$.

SUMMARY

1. The presence of backlash can lead to undesirable system excitation.

2. The excitation is rigid.

3. Even for low noise, one can cite a time interval during which the system is excited with probability arbitrarily close to unity.

In conclusion, the author wishes to thank Prof. S.P. Strelkov for the posing of the problem and for the very useful discussion of the results obtained.

LITERATURE CITED

[1] R. L. Stratonovich, "Self-excited oscillator synchronization in the presence of noise," Radiotekh. i Elektron. 4 (1958).

[2] R. L. Stratonovich and R. S. Landa, "Noise stimulus on generators with rigid excitation," Izvest. Vysshikh Uchebn. Zavedenii, Ser. Radiofizika 1 (1959).

ON UNBIASED ESTIMATES OF SIGNALS WHICH DEPEND NONLINEARLY ON UNKNOWN PARAMETERS

I. A. Boguslavskii

(Moscow)

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A method is presented which, in many cases, allows one to use linear theory for obtaining, in the presence of noise, an unbiased estimate of a signal which depends nonlinearly on unknown parameters.

In [1-4], the problem of obtaining, in the presence of noise, an optimal, in some sense or other, estimate of a signal which is a linear function of unknown parameters was solved. In many cases of practical importance, the signal is a nonlinear function of these parameters. The general solution of the corresponding nonlinear problem was obtained by V. S. Pugachev.*

However, there is a definite practical interest in artificial ways of effectively solving the problem in certain cases by using the well-known linear methods of statistical dynamics. One such method is presented below. The method allows one, for example, to implement sufficiently simply an unbiased filtering of a random process whose signal satisfies a nonlinear differential equation of a definite type with unknown initial conditions. The method can be used for smoothing, without dynamic errors, the coordinates and velocity of Sputniks and cosmic rockets during their unpowered flight phases, etc.

Description of the Method

A random process $Z(t)$ is observed, its form being

$$Z(t) = \Phi(c_1, c_2, \ldots, c_n, t) + m(t),$$

where Φ is the signal, the c_k are random parameters, and $m(t)$ is the noise.

Let it be possible to give a transformation I which takes Φ to a function of the form $\sum_{k=1}^{n} c_k W_k(t)$ with the $W_k(t)$ known a priori, and a transformation II which, when applied to the result of an application of transformation I, leads back to the function Φ. In addition, we shall assume that the random process $Z_1(t)$ which results from the application of transformation I to $Z(t)$, may be presented approximately in the form

$$Z_1(t) = \sum_{k=1}^{n} c_k W_k(t) + m_1(t),$$

where $m_1(t)$ is a random noise process whose correlation function can be expressed in terms of the probability characteristics of the noise $m(t)$.

Then, a scheme for obtaining an unbiased estimate of the signal can be constructed in the following manner.

1. We apply transformation I to $Z(t)$.
2. We apply $Z_1(t)$ to the input of an optimal unbiased filtering link, constructed in accordance with the well-known linear methods on the basis of the data on the functions $W_k(t)$ and the correlation function of the noise $m_1(t)$; after the selected memory time T of the link, we obtain at its output the function $\sum_{k=1}^{n} c_k W_k(t)$ with a decreased random error.
3. We apply transformation II to the output of this link.

The realization of the sequence of operations just described allows one to construct, with time lag T, the function $\Phi(c_1, c_2, \ldots, c_n, t)$ without dynamic errors and with decreased random errors only if transformation II insignificantly "increases" the noise at the output of the optimal linear link.

Let it be known that the signal Φ satisfies a nonlinear differential equation of the form

$$a_0(t)\frac{d^n\Phi}{dt^n} + a_1(t)\frac{d^{n-1}\Phi}{dt^{n-1}} + \cdots + a_n(t)\Phi = F[t, \Phi] \quad (1)$$

with unknown initial conditions

$$c_1 = \Phi(0), \quad c_2 = \Phi^{(1)}(0), \ldots, c_n = \Phi^{(n-1)}(0).$$

Because of the presence of the function $F(t, \Phi)$ in the right member of (1), the signal in the case under consideration is a nonlinear function of the unknown parameters $c_1, c_2, \ldots, c_n$. It follows from (1) that

$$\Phi(c_1, c_2, \ldots, c_n, t) = \sum_{k=1}^{n} c_k W_k(t) + \int_0^t W(t, \tau) F[\tau, \Phi]\, d\tau, \quad (2)$$

where $W_k(t)$ is the solution of the homogeneous equation

$$a_0(t)\frac{d^n W_k}{dt^n} + a_1(t)\frac{d^{n-1}W_k}{dt^{n-1}} + \cdots + a_n(t) W_k = 0$$

with initial conditions

$$W_k^{(m)}(0) = \begin{cases} 0 & \text{for } m \neq k, \\ 1 & \text{for } m = k, \end{cases}$$

and $W(t, \tau)$ is the impulsive response of the corresponding linear dynamic link.

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* In a paper given February 16 and March 2, 1959, at the seminar on probabilistic methods in automatic control theory of the Institute of Automation and Remote Control of the Academy of Sciences of the USSR.

It is clear from (2) that transformation I should be defined by the relationship

$$Z_1(t) = Z(t) - \int_0^t W(t, \tau) F[\tau, Z(\tau)]\, d\tau, \quad (3)$$

and transformation II by the relationship

$$Z_2(t) = Z^*(t) + \int_0^t W(t, \tau) F[\tau, Z(\tau)]\, d\tau, \quad (4)$$

where $Z^*(t)$ denotes the result of optimal unbiased filtering of the process $Z_1(t)$. After time T, given by the solution of the optimization problem, $Z_2(t)$ will differ from $\Phi(c_1, c_2, \ldots, c_n, t)$ only by a random error.

As a rule, one can approximately set $m_1(t) \approx m(t)$ only if the link with impulsive response $W(t, \tau)$ possesses essential filtering properties, and $F[t, \Phi]$ is of the same order as Φ. Analogously to what has been presented, one can obtain an unbiased estimate of the result of applying any given linear operator S to the function Φ.

In this case

$$Z_2(t) = Z^*(t) + S\left\{\int_0^t W(t, \tau) F[\tau, Z(\tau)]\, d\tau\right\},$$

where $Z^*(t)$ is the optimal unbiased estimate, after time lag T, of the function $S\left\{\sum_{k=1}^{n} c_k W_k(t)\right\}$.

We note that, for a given function $\Phi(c_1, c_2, \ldots, c_n, t)$, the corresponding nonlinear differential equation can be found by eliminating the parameters c_k from the following system of n + 1 algebraic equations:

$$\Phi = \Phi(c_1, c_2, \ldots, c_n, t),$$
$$\frac{d\Phi}{dt} = \Phi_t'(c_1, c_2, \ldots, c_n, t),$$
$$\frac{d^n\Phi}{dt^n} = \Phi_t^{(n)}(c_1, c_2, \ldots, c_n, t).$$

However, a direct application of the transformations of (3) and (4) can only be used for obtaining an unbiased estimate in the case when the nonlinear portion of the differential equation obtained depends only on Φ and t.

By using the Jacobian of the functions $\Phi, \Phi_t', \ldots$ with respect to $c_1, c_2, \ldots$, one can obtain an analytic expression for the condition cited.

For n = 1, this condition is always met. Thus, if for example,

$$\Phi = \frac{c}{c+t},$$

then

$$\frac{d\Phi}{dt} = -\frac{c}{(c+t)^2} \text{ and } t\frac{d\Phi}{dt} + \Phi = \Phi^2.$$

If, to the method presented, one adds the optimally determined derivatives of the function $Z_1(t)$, one can then, in conjunction with the method of successive approximation, obtain an unbiased estimate even when the nonlinear portion is a function of the derivatives of the function Φ.

Obtaining an unbiased estimate by the scheme presented above does not require that a nonlinear differential equation be solved. Therefore, in the example to be considered below, the use of the scheme allows a concrete engineering problem to be solved in an uncomplicated way.

Determination of the Smoothed Values of the Elements of a Sputnik's Trajectory

We now construct a methodology for determining, without dynamic errors, the smoothed values of the position and velocity of a Sputnik. For brevity we shall consider the problem in a plane; taking nonplanar motion into account for the smoothing engenders no difficulties.

Let x and y be the coordinates of the Sputnik with respect to a rectangular coordinate system passing initially through the center of the earth but not rotating with it. If one neglects the atmospheric braking of the Sputnik and the effect of the nonsphericity of the earth, then x and y satisfy the nonlinear differential equations

$$\ddot{x} = -gR^2 \frac{x}{(x^2 + y^2)^{\frac{3}{2}}}, \quad (5)$$

$$\ddot{y} = -gR^2 \frac{y}{(x^2 + y^2)^{\frac{3}{2}}}, \quad (6)$$

where R is the radius of the earth; g = 9.81 meters/sec^2.

Therefore, x(t), for example, is described by the formula

$$x(t) = x(0) + \dot{x}(0)\,t - \int_0^t \frac{gR^2 (t-\tau)\, x(\tau)}{[x^2(\tau) + y^2(\tau)]^{\frac{3}{2}}}\, d\tau \quad (7)$$

or by the formula

$$x(t) = x(0)\cos\omega t + \frac{\dot{x}(0)}{\omega}\sin\omega t -$$
$$-\frac{1}{\omega}\int_0^t \sin\omega(t-\tau)\, x(\tau)\left\{\frac{1}{[x^2(\tau) + y^2(\tau)]^{\frac{3}{2}}} - \frac{1}{R^3}\right\} d\tau, \quad (8)$$

where $\omega = \sqrt{g/R}$.

In formula (8) the portion of the signal which depends linearly on the nonlinear parameters x(0) and $\dot{x}(0)$ is separated more completely than in formula (7). However, the effectiveness of the filtering turns out to be practically the same, whether (7) is used or (8). We may therefore use formula (7) which provides a simpler form of the linear operator which implements optimal unbiased filtering of the result of transformation I. Let the following random process be observed

$$Z_x(t) = x(t) + m_x(t),$$

where $m_x(t)$ is the random error in measuring the coordinate $x(t)$ by terrestrial radar (radio) and optical means.

It follows from (7) that transformation I is defined by the relationship

$$Z_{x1}(t) = Z_x(t) + \int_0^t \frac{gR^2(t-\tau)Z_x(\tau)}{[Z_x^2(\tau)+Z_y^2(\tau)]^{\frac{3}{2}}}\,d\tau, \quad (9)$$

and transformation II by the relationship

$$Z_{x2}(t) = Z_x^{*}(t) - \int_0^t \frac{gR^2(t-\tau)Z_x(\tau)}{[Z_x^2(\tau)+Z_y^2(\tau)]^{\frac{3}{2}}}\,d\tau. \quad (10)$$

The nub of the problem under consideration is that, in the given case, the function $W(t, \tau)$ does not possess the filtering properties [with the use of (7) or (8), $W(t, \tau)$ equals, respectively, $t-\tau$ or $\sin\omega(t-\tau)/\omega$]. Therefore, the dispersion of the noise $m_{x1}(t)$ contained in $Z_{x1}(t)$ increases without bound in the course of time.

However, if T is limited by some value completely admissible for practical purposes (cf. the Appendix), then $m_{x1}(t)$ coincides in practice with the noise $m_x(t)$. Therefore, with the known correlation function of the noise $m_x(t)$, it is easy to construct a scheme which takes $Z_{x1}(t)$ to $Z_x^{*}(t)$, which is the optimal unbiased filtering of the random process whose signal is a linear function of time.

In the case of the limitation on T already mentioned, transformation II does not in practice decrease the efficiency of this filtering.

In an analogous way, one carries out the unbiased filtering of the random errors arising in the measurement of coordinate $\underline{y}$. If it is necessary to find $V_x(t)$ and $V_y(t)$, the projections of the Sputnik's velocity vector, then transformation I, as before, is described by relationship (9), and transformation II takes the form

$$V_x(t) = V_x^{*}(t) - \int_0^t \frac{gR^2 Z_x(\tau)}{[Z_x^2(\tau)+Z_y^2(\tau)]^{\frac{3}{2}}}\,d\tau,$$

where $V_x^{*}(t)$ is the result of unbiased optimal differentiation of random process $Z_{x1}(t)$.

If the lack of sphericity of the earth is taken into account, the smoothing scheme remains unchanged only if the constant $\underline{g}$ in the formula is replaced by a function of $\underline{x}$ and $\underline{y}$, defined by well-known relationships of potential theory. To take into account the braking of the Sputnik by the atmosphere, one must add, to the right members of Eqs.(5) and (6) and, certainly, to the integrands in (9) and (10), terms which depend, not only on $\underline{x}$ and $\underline{y}$, but also on V_x and V_y. The forms of these terms are known if we know the density of the atmosphere as a function of $\underline{x}$ and $\underline{y}$, and the drag coefficient as a function of V. With this, the smoothing method is somewhat complicated, it being required to use successive approximations, and may be laid out in the following way. First, using the method given above, wherein the braking force is not taken into account, we find V_x and V_y with some dynamic error. The values of V_x and V_y found are used to construct terms in which this force is taken into account, these terms being added to the integrands in (9) and (10), after which the process is repeated, resulting in new, more accurate values of V_x and V_y. By carrying out this process several times, we obtain V_x and V_y practically without dynamic errors, and with random errors defined, for limited T, on the basis of the random errors of the optimal differentiation. We may note that frequently one need not use successive approximations but may, instead of V_x and V_y, use the results of numerical differentiation with specially chosen increments of the observed quantities Z_x and Z_y.

The method can also be used, without change, for obtaining smoothed values of the coordinates and the projections of the velocity of cosmic rockets (space probes) describing complicated paths under the influence of the gravitational fields of earth and several other heavenly bodies. In this case, the right members of Eqs. (5) and (6) contain terms which depend on $\underline{x}$, $\underline{y}$ and on the current value of $\underline{t}$, these terms taking into account the variations in the coordinates of these bodies with respect to the earth during the course of the smoothing process.

APPENDIX

We denote by $\Delta(t)$ the random component contained in the integral terms of formulas (9) and (10) due to the presence of the random errors $m_x(t)$ and $m_y(t)$ and, by $\sigma_\Delta^2(t)$, we denote its dispersion.

We assume that, in practice, the magnitude of $x^2 + y^2$ is several orders of magnitude greater than the quantity $2(xm_x + ym_y) + m_x^2 + m_y^2$. Thus,

$$\Delta(t) \approx \int_0^t \frac{gR^2(t-\tau)m_x(\tau)}{[x^2(\tau)+y^2(\tau)]^{\frac{3}{2}}}\,d\tau$$

and

$$\sigma_\Delta^2(t) \leqslant \frac{g^2}{R^2}\int_0^t\int_0^t (t-\tau_1)(t-\tau_2)\,\varphi_x(\tau_1,\tau_2)\,d\tau_1\,d\tau_2, \quad (11)$$

where $\varphi_x(t_1, t_2)$ is the correlation function of random error $m_x(t)$.

We assume that

$$\varphi_x(t_1, t_2) = \sigma_0^2 e^{-\gamma|t_2-t_1|}.$$

It follows from (11) that, in this case

$$\sigma_\Delta(t) \leqslant \sigma_0 \frac{g}{R}\sqrt{\frac{2}{3\gamma}t^3 - \frac{1}{\gamma^2}t^2 - \frac{2}{\gamma^3}te^{-\gamma t} + \frac{2}{\gamma^4}(1-e^{-\gamma t})}. \quad (12)$$

For definiteness, we set $\gamma = 1\ \mathrm{sec}^{-1}$, and we shall assume that, in constructing the scheme for the optimal

unbiased filtering which takes $Z_{x1}(t)$ to $Z_x(t)$, and for computing σ_x^2, the dispersion of the noise at the output of this scheme, we can ignore $\Delta(t)$ ($m_x(t) \approx m_{x1}(t)$), if

$$\sigma_\Delta(t) < 0{,}1\ \sigma_0. \tag{13}$$

It follows from (12) that (13) holds for $t < 1900$ sec. In this case, the optimal circuit's impulsive response is easily found, and

$$\sigma_x^2 = \sigma_0^2 \frac{24 + 24\gamma T + 8\gamma^2 T^2}{24 + 24\gamma T + 8\gamma^2 T^2 + \gamma^3 T^3}, \tag{14}$$

where T is the memory of the optimal circuit and σ_x^2 is the dispersion of the noise contained in $Z_x^*(t)$.

It follows from (12) and (14) that the addition of an integral term in correspondence with formula (10) to the output of the optimal circuit does not, in practice, increase the random error, if $t = T < 500$ sec. Therefore, the memory of the optimal circuit should be defined by the conditions

$$T = t \quad \text{for} \quad t < 500 \text{ sec},$$
$$T = 500 \text{ sec} \quad \text{for} \quad t \geqslant 500 \text{ sec}.$$

In this case, formula (14) can be used to define the dispersion of the random error in the smoothed coordinates of the Sputnik.

Starting with time $t = 500$ sec, the lower limit of the integrals in the right members of formulas (9) and (10) should be replaced by $t-500$.

It is shown analogously that this same choice of T allows one to neglect $\Delta(t)$ in the determination of V_x also. Then, the impulsive response of the optimal differentiating circuit is easily found, and σ_V^2, the dispersion of the random error in the determination of the projections of the Sputnik's velocity, is described by the relationship

$$\sigma_V^2 = \sigma_0^2 24\gamma^3 \frac{2 + \gamma T}{24\gamma T + 24\gamma^2 T^2 + 8\gamma^3 T^3 + \gamma^4 T^4}. \tag{15}$$

If, for example, we choose $\sigma_0 = 1$ km, we get from (14) and (15) that, after 200 sec of receiving information from the Sputnik, one can find its coordinates and the projections of its velocity without dynamic error and with random errors characterized by the quantities $\sigma_x = 200$ m and $\sigma_V = 1.7$ m/sec.

LITERATURE CITED

[1] V. S. Pugachev, Theory of Random Functions and Their Applications to Problems of Automatic Control (Gostekhizdat, 1957).

[2] V. S. Pugachev, "The determination of optimal systems by arbitrary criteria," Avtomat. i Telemekh. 19, 6 (1958).†

[3] V. V. Solodovnikov, Introduction to the Statistical Dynamics of Automatic Control Systems [in Russian] (Gostekhizdat, 1952).

[4] J. H. Laning and R. H. Battin, Random Processes in Automatic Control [Russian translation] (IL, 1958).

† See English translation.

SYNTHESIZING CONTROL SYSTEMS WITH MONOTONICALLY DECREASING GAINS BY THE ROOT-LOCUS METHOD

Ya. I. Rubinovich

(Moscow)

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We consider a periodically connected control system which contains a link whose gain decreases monotonically with time. We introduce the concept for the quasi-majorant closed control system for which, using the root-locus method, we choose a correcting circuit.

Posing of the Problem

We assume that the control process lasts for time T, after which the control system returns to its initial state, and the process is again repeated when the controlled object is subjected to the same, or to an analogous, perturbing movement.

In the general case, the open-loop control circuit consists of two series-connected links: link I with transfer function

$$G(s) = \frac{\prod\limits_{i=1}^{m} (s + z_i)}{\prod\limits_{i=1}^{n} (s + p_i)}, \tag{1}$$

where $n \geq m$, and the p_i lie in the left half of the s plane, and link II, whose variable gain decreases monotonically with time in accordance with the law

$$K_1(t) = \frac{K_0}{(t + a)^r}. \tag{2}$$

The quantity r lies within the limits $1 \leq r \leq 4$, and depends on the telecontrol method being used.

To simplify the analysis, we replace link II by link II*, whose gain varies in accordance with the law

$$K_1^*(t) = Ae^{-\frac{t}{\tau}}. \tag{3}$$

The coefficients A and τ are so chosen than $K_1(0) = K_1^*(0)$ and $K_1(T) = K_1^*(T)$. Then

$$A = \frac{K_0}{a^r}, \qquad \tau = \frac{T}{\ln\left(1 + \frac{T}{a}\right)^r}$$

and

$$K_1^*(t) = \frac{K_0}{a^r} \exp\left[-\frac{t}{T} r \ln\left(1 + \frac{T}{a}\right)\right]. \tag{4}$$

It follows, from the properties of exponential functions, that

$$K_1^*(t) > K_1(T) \text{ for } 0 < t < T.$$

The system thus obtained, consisting of links I and II*, will be quasi-majorant* with respect to the initial system with link II.

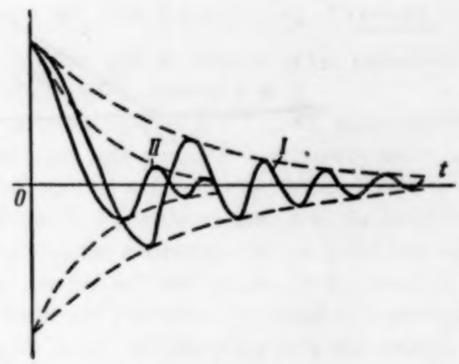

Fig. 1

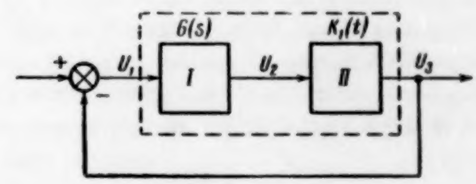

Fig. 2

Since function G(s) has no poles in the right half-plane, the closed-loop system with link II* (Fig. 2) will also be quasi-majorant with respect to the closed-loop system with link II.

If a signal $U_1(t)$ is applied to the input of link I of the open-loop system then, at the output of link II*, one obtains the signal $U_3(t)$, whose Laplace transform has the form:

$$U_3(s) = AG\left(s + \frac{1}{\tau}\right) U_1\left(s + \frac{1}{\tau}\right). \tag{5}$$

Thus, the exponentially decreasing gain is equivalent, in the open-loop system, to a shift to the left by the amount $1/\tau$ of all the zeros and poles of the transfer function and of the disturbing stimulus. It is obvious that, if the zeros and poles of the disturbing stimulus are

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* A system will be called quasi-majorant with respect to another system if the envelope of the first system's transient response completely contains the transient response curve of the second (given) system. For example, the system which gives the transient response corresponding to curve I on Fig. 1 is quasi-majorant with respect to the system giving curve II for its transient response.

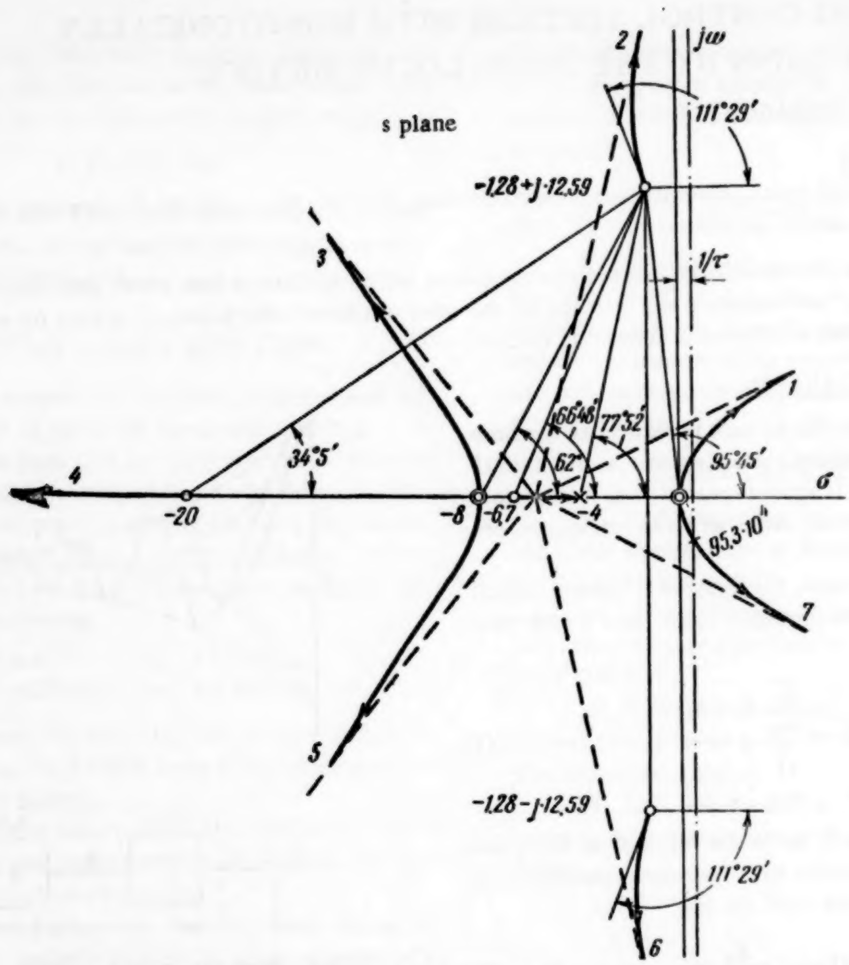

Fig. 3. O is a simple pole, ⊚ is a double pole, and × is a simple zero.

shifted to the right by the same quantity, $1/\tau$, then the curve of $U_3^*(t)$ which one obtains in this case will be quasi-majorant with respect to curve $U_3(t)$ corresponding to Eq. (5). The system with open-loop transfer function $AG(s + 1/\tau)$ will, after closing the loop by a feedback path, be quasi-majorant with respect to the initial system (Fig. 2). Therefore, one can make certain estimates of stability and transient response quality for the initial system by investigating the quasi-majorant system obtained.

We use the following concrete example for showing the method of synthesizing such control systems.

Let link I's transfer function have the following form:

$$G(s) = \frac{(s + z_1)}{s^2 (s + p_1)(s + p_2)^2 (s + p_3)(s^2 + 2\zeta\, \omega_n s + \omega_n^2)}, \tag{6}$$

where $z_1 = 4$, $p_1 = 6.7$, $p_2 = 8$, $p_3 = 20$, $\zeta = 0.101$ and $\omega_n = 12.65/\text{sec}$.

The gain of link II varies in accordance with the following law:

$$K_1(t) = \frac{K_0}{t + a}, \tag{7}$$

where $K_0 = 2 \cdot 10^5$ and $a = 0.2$ sec. Operation time is $T = 10$ sec.

It is necessary to so choose a series-connected correcting circuit that, after closing the loop with a feedback path, one obtains a relative damping coefficient† for the system of not less than 0.2 for any pair of complex poles.

Root Motion Analysis of the Closed-Loop System

We draw the positions of the zeros and poles of function $G(s)$ on the complex $s = (\sigma + j\omega)$ plane (Fig. 3). The arrows on the curves show the direction of motion of the poles as K_{syst} increases.

If the system is closed by a negative feedback path (Fig. 2) then, as the system gain K_{syst} increases from zero to infinity, the roots of the closed-loop system's characteristic equation (the poles of the closed-loop system) will move from the poles of the open-loop system to its zeros.

We now construct the trajectories of the roots as they move. The rules for constructing such trajectories have been published in the literature [1-5], so that we shall use them here without proof. As is well known, the

† By the relative damping coefficient we understand the damping coefficient which corresponds to the presence of just one pair of complex poles.

number of individual trajectory branches tending to infinity equals the difference

q = (number of finite poles) –
– (number of finite zeros).

In our example, there will be seven such branches. For a sufficiently large gain K, these trajectory arms tend to asymptotes which start at one point on the real axis and form, with that axis, angles equal to $\pm [180°/q + i \cdot 360°/q]$, where i = 0, 1, 2, 3, . . . , etc.

We find the coordinate of the point of intersection of the asymptotes by the formula

$$s_1 = \frac{\Sigma(\text{ poles }) - \Sigma(\text{zeros})}{q} =$$

$$= \frac{(-6.7 - 2.8 - 20 - 2 \cdot 1.28) - (-4)}{7} = -5.89$$

and we draw the asymptotes at the angles (Fig. 3)

$$\theta_1 = \pm \frac{180°}{7} = \pm 25°42',$$

$$\theta_3 = \pm\left(25°42' + \frac{360°}{7} \cdot 2\right) = \pm 128°30',$$

$$\theta_2 = \pm\left(25°42' + \frac{360°}{7}\right) = \pm 77°6',$$

$$\theta_4 = \pm\left(25°42' + \frac{360°}{7} \cdot 3\right) = \pm 180°.$$

All portions of the real axis lying to the left of an odd-numbered root (zero or pole), where the roots are counted starting from the far right, are parts of trajectories. We thus find that the pole at point s = –6.7 moves along the real axis to the zero at point s = –4, and the pole at point s = –20 tends to infinity along the real axis towards the left.

We now determine the slope of the trajectory starting from the pole s = –1.28 + 12.59j. To do this, we join this pole with the remaining roots and poles. The slope angle sought, θ, is found from the relationship

Σarg(vectors from zeros) – Σarg(vectors from poles) – θ° = 180° ± n · 360°.

In our case,

$$77°52' - [90° + 2 \cdot 95°45' + 66°48' +$$

$$+ 2 \cdot 62° + 34°5'] - \theta° = 180° + n \cdot 360°,$$

$$\theta° = -428°31' - 180° \pm n \cdot 360° = -248°31'$$

$$\text{or} \quad -111°29'.$$

We can now easily construct approximate trajectories of the roots' motion, as was done on Fig. 3. If all the curves are shifted to the left by the amount

$$\frac{1}{\tau} = \frac{\ln\left(1 + \frac{T}{a}\right)}{T} = \frac{\ln 51}{10} = 0.393$$

or, equivalently, the imaginary axis is shifted to the right by this amount, we obtain the root locus of the quasi-majorant system (Fig. 3).

A system with a constant gain will be unstable for any value of that gain, since the double pole at the origin will immediately enter into the right half-plane.

The quasi-majorant system becomes unstable for $K_{syst} \geq 95.3 \cdot 10^4$, as one may verify by determining the value of the gain for which the pole moving along arm I intersects the shifted imaginary axis. For this, we take the product of the lengths of the vectors from the poles to the point of intersection and divide this by the product of the lengths of the vectors from the zeros to this same point.

Choice of the Correcting Circuit

We first give the choice of the correcting circuit for a system with a constant gain.

In order to keep arm I of the trajectory from entering the right half-plane, it is necessary to introduce a zero on the real axis on the segment from 0 to s = – 6.7. The closer this additional zero is to the origin, the more sharply will the trajectories of the poles turn up from the origin into the left half-plane. In the limiting case, the introduction of a zero at the origin will neutralize one of the poles there, and the other pole will tend to the zero at the point s = – 4 along the real axis. The introduction of a zero at the origin means the introduction of a pure derivative of the signal, which can be accomplished in practice either by means of a special differentiating amplifier circuit, or by the use of tachometer generators in the cases where the circuit's input signal is in the form of a shaft rotation.

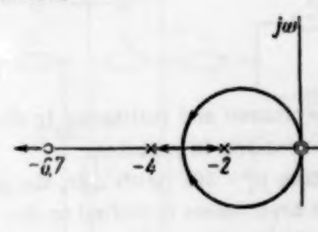

Fig. 4

For the realization of a zero on the real axis by a passive RC circuit, it is desirable that the zero not be too close to the origin. We shall introduce a zero at the point $s = -z_1^* = -2$. Then, an approximate sketch of the motion of the root in the neighborhood of the origin takes the form shown in Fig. 4.

For realizing this zero by means of a series-connected passive RC network, it is necessary that the transfer function of the correcting circuit have the form:

$$G_2(s) = \frac{(s + z_1^*)}{(s + p_1^*)} \frac{p_1}{z_1^*}.$$

It is desirable to choose the pole p_1^* sufficiently far from the origin so that the component of system behavior attributable to it will be rapidly damped. However, it is undesirable to remove it too far, since the

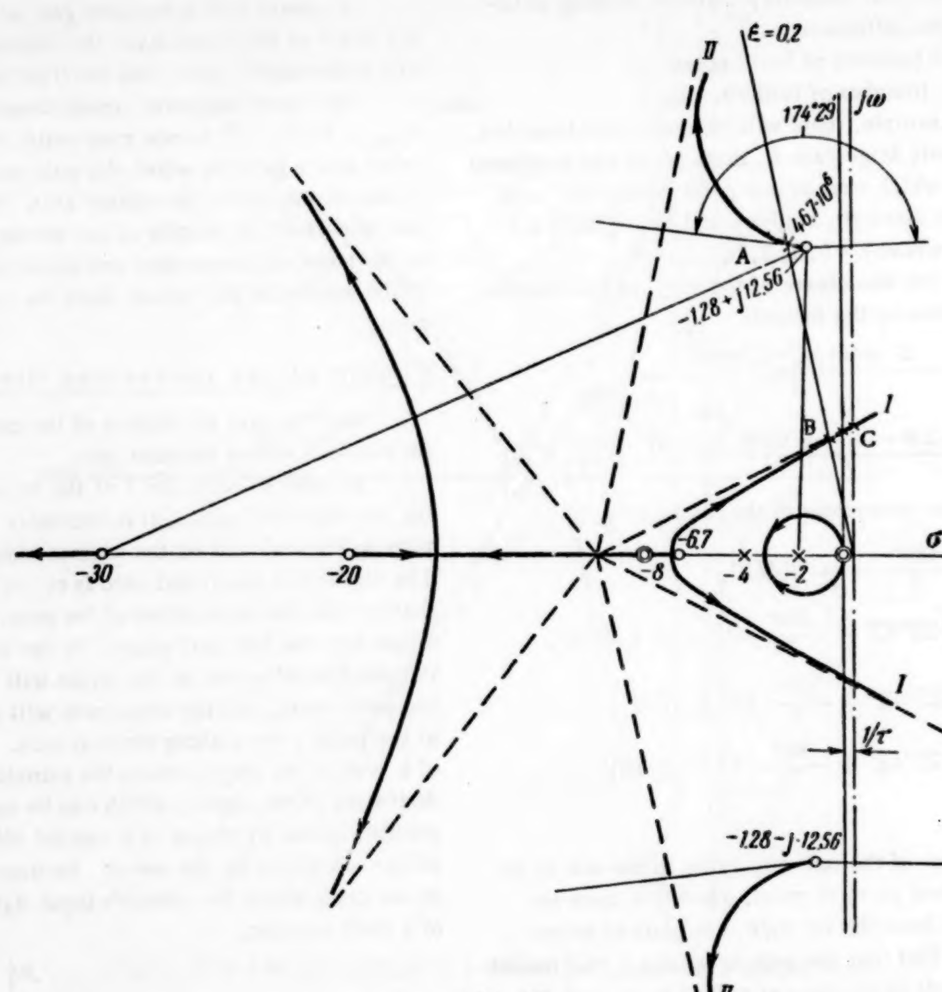

Fig. 5

requisite capacitance and resistance in the network would be difficult to realize in practice.

We choose $p_1^* = 30$. With this, the point of intersection of the asymptotes is shifted to the left by the amount $(p_1^* - z_1^*)/7 = (30-2)/7 = 4$.

The slopes of the asymptotes do not change. By again joining the pole and zero introduced with the point $s = -1.28 + 12.59j$, we find the additional rotation of the trajectory starting from the pole:

$\Delta\theta^* = \arg(\text{vector from } z_1^*) - \arg(\text{vector from } p_1^*) = 86°45' - 23°45' = 63°$.

The root locus takes the form shown on Fig. 5. We now determine for what gain the poles of the closed-loop quasi-majorant system will lie on the intersections of the line of constant damping coefficient, $\xi = 0.2$, with trajectories I and II.

For point A we get

$$K_{\text{syst}} = \frac{30.6\cdot 22\cdot 14^2\cdot 13.6\cdot 13.1^2\cdot 25\cdot 4\cdot 1}{13\cdot 12.9} = 46.7\cdot 10^6;$$

for point B

$$K_{\text{syst}} = \frac{29.8\cdot 20\cdot 8.72^2\cdot 7.6\cdot 4.7^2\cdot 8.1\cdot 9.2}{5.8\cdot 4.8} = 20.5\cdot 10^6.$$

At the boundary of stability of the quasi-majorant system (point C)

$$K_{\text{syst}} = \frac{30.8\cdot 21\cdot 9.8^2\cdot 8.7\cdot 5^2\cdot 17.7\cdot 7.8}{6.8\cdot 5.6} = 49\cdot 10^3.$$

For the quasi-majorant system, the required gain is

$$K_{\text{syst}} = \frac{p_1^*}{z_1^*}\,\frac{K_0}{a} = \frac{30}{2}\,\frac{2\cdot 10^5}{0.2} = 15\cdot 10^6.$$

Thus, the introduction of the correcting circuit provides system stability, but does not give the necessary damping coefficient.

We note that the introduction of zeros and poles on the real axis does not, in general, have much effect on the displacement of the complex poles when the gain is small, since they will be translated in a small neighborhood of the point $s = -1.28 \pm 12.59j$. Consequently, the sole means of obtaining the necessary damping coefficient is to eliminate these poles, which can be done by a correcting circuit with zeros at the points cited.

We thus obtain the following desirable transfer function for the correcting circuit:

$$G_2(s) = \frac{(s+2)\,(s^2 + 2\xi\omega_n s + \omega_n^2)\,15p_2^* p_3^*}{(s+30)\,(s+p_2^*)\,(s+p_3^*)\,\omega_n^2}.$$

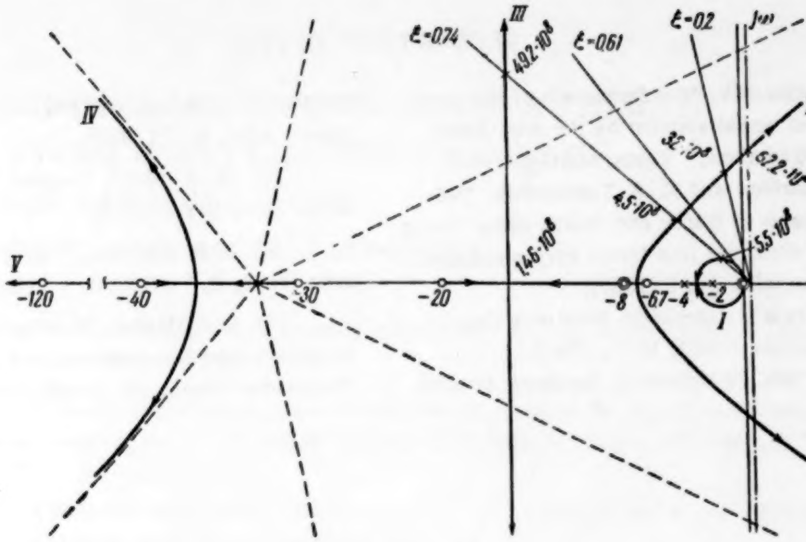

Fig. 6.

The poles at the points $-p_2^*$ and $-p_3^*$ are introduced in order to obtain a passive network realizable by capacitances and resistances. As a rule, these should be chosen further to the left than the original zeros and poles, in order that the components in the transient response introduced by them be rapidly damped. In addition, the introduction of these poles moves the point of intersection of the asymptotes to the left, which increases the system's stability.

We choose $p_2^* = 40$ and $p_3^* = 120$. Then, the open-loop circuit thus obtained will have the following transfer function:

$$G_{\text{syst}}(s) =$$

$$= \frac{K_{\text{syst}}\ (s+2)(s+4)}{s^2(s+6.7)(s+8)^2(s+20)(s+30)(s+40)(s+120)}.$$

The magnitude of K_{syst} for the quasi-majorant system (with the correcting circuit) will equal

$$K_{\text{syst}} = \frac{15\dot{p}_2\dot{p}_3}{\omega_n^2}\frac{K_0}{a} = \frac{15\cdot 40\cdot 120}{12.65^2}\cdot\frac{2\cdot 10^6}{0.2} = 4.5\cdot 10^8.$$

The root locus of the closed-loop system is shown on Fig. 6. The values of the gains are given directly on the curves. The relative damping coefficient of any pair of complex poles is greater than 0.2. With this, the pair of poles on arm III of the trajectory provides a component of the transient response which is rapidly damped and has small oscillation. The poles on arm I provide a component which is small in magnitude (due to the proximity of the zeros at points -2 and -4) and which is slowly damped.

The roots on arms IV and V give very rapidly damped terms and do not have any practical effect on the system's transient response or frequency characteristic. Thus, the system obtained completely satisfies the requirements posed.

When the poles $-1.28 \pm 12.59j$ are not exactly compensated, there arises a slowly damped, rapidly oscillating, component of insignificant amplitude whose magnitude equals the residues at these poles, i.e., is proportional to the magnitude of the dipole obtained in this case [3, 5].

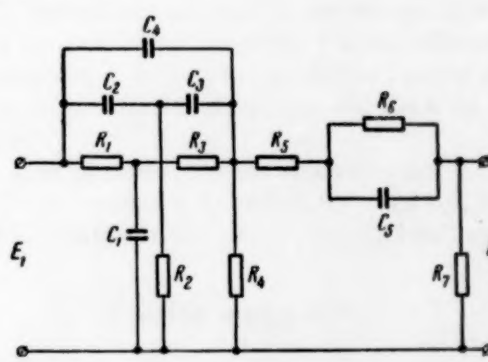

Fig. 7. $R_1 = 11.6$ kohm, $R_2 = 1.1$ kohm, $R_3 = 23$ kohm, $R_4 = 11.3$ kohm, $R_5 = 22.5$ kohm, $R_6 = 832$ kohm, $R_7 = 17.7$ kohm, $C_1 = 2.5\ \mu$f, $C_2 = 22\ \mu$f, $C_3 = 1.1\ \mu$f, $C_4 = 0.0725\ \mu$f, $C_5 = 0.6\ \mu$f.

The correcting function is implemented by the network shown in Fig. 7. The synthesis of this network from the given function $G_2(s)$ was carried out by Dasher's method [7]. To connect the network in the circuit, it is necessary to equalize the gains of the network and of the function $G_2(s)$ since, for dc, $G_2(0) = 1$ and the network gives $E_2/E_1 = 0.005$. Consequently, it is necessary to connect in the circuit an amplifier with gain equal to

$$K_a = \frac{1}{0.005} = 200.$$

LITERATURF CITED

[1] K. F. Teodorchik, "Fundamentals of the analysis and correction of linear systems by the root-locus (natural frequencies) method," Vestn. MGU 4 (1957).

[2] G. A. Bendrikov and K. F. Teodorchik, "Migration laws of the roots of third- and fourth-order linear algebraic equations when the free terms vary continuously," Avtomat. i Telemekh. 16, 3 (1955).

[3] John G. Truxal, Automatic Feedback Control System Synthesis (McGraw-Hill, N.Y., 1955).

[4] V. C. M. Yeh, "Synthesis of feedback control systems by gain-contour and root-contour methods," Trans. AIEE, p. II (1956).

[5] W. R. Evans, Control-System Dynamics. (McGraw-Hill, N.Y., 1954).

[6] S. P. Strelkov, "On the general theory of linear amplifiers, II," Avtomat. i Telemekh. 10, 4 (1949).

[7] B. J. Dasher, "Synthesis of RC transfer functions as unbalanced two terminal pair networks," Trans. IRE Professional Group on Circuit Theory (1952).

CONSTRUCTING OPTIMAL SECOND-ORDER AUTOMATIC CONTROL SYSTEMS USING LIMITING VALUES OF CONTROL LOOP ELEMENT GAINS

S. V. Emel'yanov and A. I. Fedotova

(Moscow)

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The possibility is considered of obtaining optimal control processes in second-order automatic control systems with the use of limiting values of control loop element gains by means of nonlinear "key"-type correcting devices.

Recently, both in domestic and foreign literature, ever greater attention has been given to the questions of improving the quality of automatic control systems by means of the introduction of nonlinear transformers (function generators) which realize desirable control laws [1-10].

A significant number of works are devoted to the synthesis and realization of nonlinear control laws which provide for the flow of optimal transient responses in automatic control systems when there are limitations imposed on one or several of the systems' coordinates [11-18]. However, the question of constructing optimal automatic control systems using limiting values of control loop element gains has occupied comparatively little space.

The purpose of the present paper is to determine the control law (and also its realization) with which one can obtain optimal transient responses in second-order linear automatic control systems using limiting values of the systems' transmission coefficient.

For the systems under consideration here, we shall understand by optimal transient responses those transient responses which, for unit jump disturbances, have no overshoot, wherein the time during which the value of the controlled coordinate is less than some previously given constant value is minimal.

1. Construction of Optimal Transient Responses of Second-Order Automatic Control Systems Using Limiting Values of Transmission Coefficients

We consider a linear second-order automatic control system (Fig. 1) whose transmission coefficient is bounded.

The equations of the system's links can be written in the following form:

the equation of the object being controlled

$$T_a\dot{x} + \rho x = -\mu + f(t); \tag{1}$$

the equation of the executive mechanism

$$T_s\dot{\mu} = \xi; \tag{2}$$

the equation of the forcing device

$$x - g(t) = \varphi; \tag{3}$$

the equation of the summing device

$$\xi = K\varphi + T\dot{\varphi}.$$

Here, $\underline{x}$ is the relative deviation of the controlled coordinate, φ is the relative change of the error signal, μ is the relative deviation of the controlling organ, $1/\rho$ is the object's static transmission coefficient, T_a/ρ is the time constant of the object of control, T_s is the integration constant of the executive mechanism, K is the gain of the error signal amplifier, T is the differentiator's time constant, ξ is the summing device's output quantity, f(t) is the external disturbance and g(t) is the forcing stimulus.

After elimination of the variables $\underline{x}$ and μ from Eqs. (1)-(3), we obtain, by setting f(t) = g(t) = 0, the differential equation of the system's free (natural) oscillations.

$$\ddot{\varphi} + 2h\dot{\varphi} + \omega_0^2\varphi = 0.$$

Here,

$$2h = \frac{\rho}{T_a} + \frac{T}{T_aT_s}, \quad \omega_0^2 = \frac{K}{T_aT_s}. \tag{4}$$

In the automatic control system under consideration, let the coefficients T_a, Ts, and ρ be some constant numbers which depend on the design peculiarities of the object and the controller, and let the gains K and T be variable within the limits

$$-K_0 \leqslant K \leqslant K_0, \quad -T_0 \leqslant T \leqslant T_0,$$

where K_0 and T_0 are the limiting values of the actual system's gains.

It is required to determine the sequence of changes of the gains K and T during the flow of the transient response which is necessary for obtaining an optimal transient response in the system for stepwise changes in the magnitude of the disturbing stimulus f(t) or of the forcing function (stimulus) g(t). In other words, it is required to determine the nonlinear control law and its realization

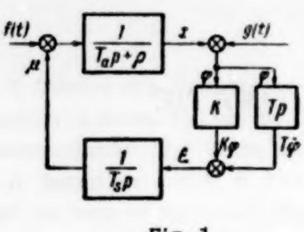

Fig. 1

for obtaining an optimal transient response in the automatic control system under investigation.

It is convenient to carry out the synthesis of a nonlinear control law for the class of systems under consideration by means of the phase plane method. Then, the problem posed is the construction of the automatic control system's phase plane which images the sequence of variations of the coefficients (structure) of the system during the transient response for various initial conditions.

The various combinations of signs of the coefficients K and T give four different system structures:

1) $K > 0, \quad T > 0;$
2) $K > 0, \quad T < 0;$
3) $K < 0, \quad T > 0;$
4) $K < 0, \quad T < 0.$

For the first case, the equation describing the free oscillations will have the form:

$$\ddot{\varphi} + 2h\dot{\varphi} + \omega_0^2\varphi = 0. \qquad (5)$$

After the elimination of time from Eq. (5), we obtain the image of the transient response on the φ, $\dot{\varphi}$ plane.

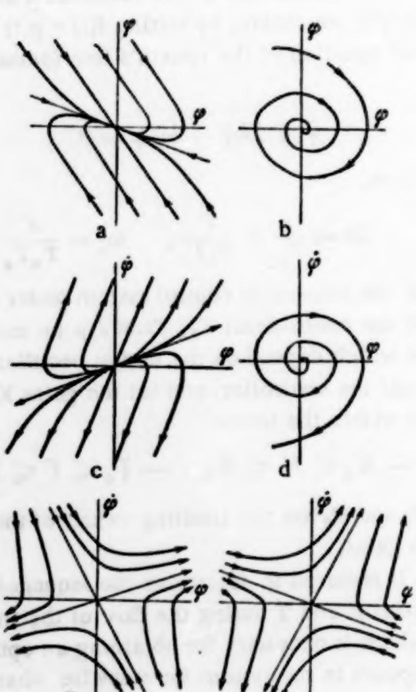

Fig. 2.

For $\omega_0^2 > h^2$, the equation of the phase trajectories will be the equation of the family of helices spiraling into the origin [6] (Fig. 2b):

$$(\dot{\varphi} + h\varphi)^2 + (\omega_0^2 - h^2)\varphi^2 = = c \exp \frac{2h}{\sqrt{\omega_0^2 - h^2}} \operatorname{arc\,tg} \frac{\dot{\varphi} + h\varphi}{\sqrt{\omega_0^2 - h^2}\,\varphi} \qquad (6)$$

where c is a constant.

For $\omega_0^2 < h^2$, the equation of the phase trajectories will be the equation of a family of deformed parabolas tangent to the line $\dot{\varphi} = -q_1\varphi$ at the origin [11] (Fig. 2a):

$$\dot{\varphi} + q_1\varphi = c_1(\dot{\varphi} + q_2\varphi)^{\frac{q_1}{q_2}}. \qquad (7)$$

Here,

$$q_1 = -h + \sqrt{h^2 - \omega_0^2},$$
$$q_2 = -h - \sqrt{h^2 - \omega_0^2},$$

and c_1 is a constant.

For the second case (if $T > \rho T_s$), the equation of the system's free oscillations will be

$$\ddot{\varphi} - 2h\dot{\varphi} + \omega_0^2\varphi = 0. \qquad (8)$$

For $\omega_0^2 > h^2$, the phase trajectories are spirals starting from the origin (Fig. 2d) and, for $\omega_0^2 < h^2$, we obtain a family of parabola-like curves on the phase plane but, as it moves along any of the integral curves, the representative (image) point will move ever further from the equilibrium position (Fig. 2c).

In the third case, the equation of the free oscillations has the form:

$$\ddot{\varphi} + 2h\dot{\varphi} - \omega_0^2\varphi = 0. \qquad (9)$$

The equation of the phase trajectories will be the equation of a family of deformed hyperbolas (Fig. 2e) [11]:

$$c_2(\dot{\varphi} - \varphi q_2')^{q_2'}(\varphi + \varphi q_1')^{q_1'} = 1. \qquad (10)$$

Here, c_2 is a constant,

$$q_1' = -h + \sqrt{h^2 + \omega_0^2}, \quad q_2' = -h - \sqrt{h^2 + \omega_0^2}.$$

The equations of the asymptotes to the hyperbolas have the form:

$$\dot{\varphi} = -q_1'\varphi, \quad \dot{\varphi} = -q_2'\varphi. \qquad (11)$$

And, for the fourth case, the equation of the system's free oscillations is written in the form

$$\ddot{\varphi} - 2h\dot{\varphi} - \omega_0^2\varphi = 0. \qquad (12)$$

The equation of the phase trajectories will also be the equation of a family of deformed hyperbolas analogous to (10). These intervals will have other slopes for

the asymptotes (Fig. 2f). For this family of curves, the asymptote equations are

$$\dot{\varphi} = -q_1^{\prime\prime}\varphi, \quad \dot{\varphi} = -q_2^{\prime\prime}\varphi. \tag{13}$$

Here,

$$q_1^{\cdot} = h + \sqrt{h^2 + \omega_0^2}, \quad q_2^{\cdot} = h - \sqrt{h^2 + \omega_0^2}.$$

We now determine the sequence of changes of the system's structure for which the image point will so move to the origin that the transient response will be optimal with respect to the controlled coordinate.

In the automatic control system under consideration, let there be applied a disturbing stimulus f(t) of the unit step type, which is equivalent to the presence of some nonzero initial conditions, specifically, $\dot{\varphi} = \varphi_0$, $\varphi = 0$. It is easily seen from a consideration of the phase trajectories (Fig. 2) that, for the image point to proceed to the equilibrium position with minimal deviation of the error φ during the duration of the transient response, with these initial conditions it must move in the region $\varphi\dot{\varphi} > 0$ along phase trajectories corresponding to an automatic control system with positive values of the coefficients K and T (Fig. 2a,b). With any other combination of signs for K and T, the image point would either move away from the equilibrium position (Fig. 2c,e,f) or have a large maximum deviation of the error φ during the transient response (Fig. 2d).

After the error signal reaches its maximum value ($\varphi = \varphi_{max}$, $\dot{\varphi} = 0$) this discrepancy should be liquidated as soon as possible. In order that this requirement be met, the phase plane for $\varphi\dot{\varphi} < 0$ must be so constructed that, for given initial conditions, the area under the integral curve will be maximal, since it is well known that, in this portion of the phase plane, the duration of the control is inversely proportional to the area under the integral curve.

In order for the response to occur without overshoot, one of the components of the integral curve to be synthesized must pass through the origin in this region of the phase plane and, in order for the area under the integral curve to be maximal, it is necessary that the slope of this phase trajectory be as large as possible.

It is clear, from a consideration of the phase sketches (Fig. 2), that there are trajectories passing through the origin in the required portion of the phase plane for three different combinations of signs of K and T (Fig. 2a,e,f). It is easily remarked that the greatest slope of the null phase trajectory corresponds to the structure of the system with a positive value of the differentiator's time constant T and a negative gain K of the error signal (Fig. 2e).

As follows from Eqs. (9) and (11), the equation of such a phase trajectory has the form: $\dot{\varphi} = -q_2^{\cdot}\varphi$. Here, $q_2^{\cdot} = -h - \sqrt{h^2 + \omega_0^2}$.

The second component of the integral curve should be so chosen that the image point will travel from position $\varphi = \varphi_{max}$, $\dot{\varphi} = 0$ along the null phase trajectory chosen by us with the largest possible value of the coordinate $\dot{\varphi}$. This also makes it possible to obtain the greatest area under the integral curve. From this point of view, those phase trajectories are most suitable which correspond to the unstable structural scheme with a positive value of K and a negative value of T (Fig. 2c and d).

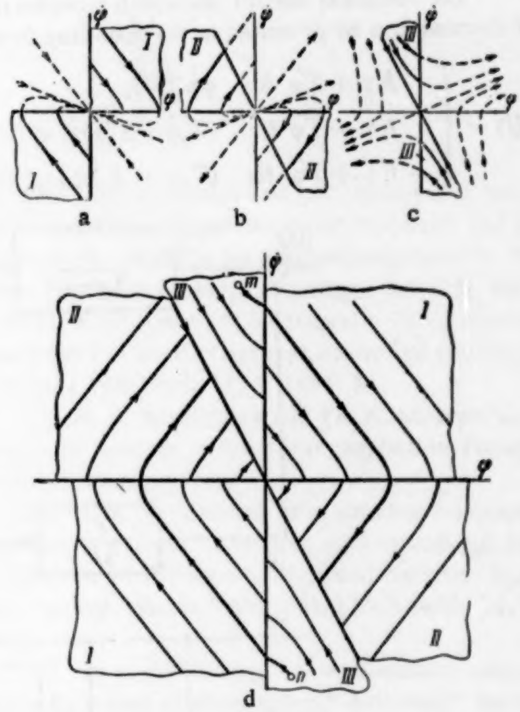
Fig. 3.

The process of constructing the phase "portrait" for obtaining an optimal transient response in the system is shown in Fig. 3a,b,c, and d.

Thus, the phase portrait of the optimal control system can be presented in the form of a three-sheeted phase surface (Fig. 3d).* The first sheet corresponds to the phase plane satisfying the condition $\varphi\dot{\varphi} \geq 0$ (Fig. 3a) and Eq. (5) for the free oscillations. The second sheet corresponds to the portion of the phase plane which satisfies the condition $(T_j\dot{\varphi} + K_j\varphi)\dot{\varphi} < 0$ (Fig. 3b).

The constant coefficients T_j and K_j are so chosen that the boundary of the sheet, corresponding to the equation

$$T_j\dot{\varphi} + K_j\varphi = 0,$$

coincides with the phase trajectory described by the equation $\dot{\varphi} = -q_2^{\cdot}\varphi$, where $q_2^{\cdot} = -h - \sqrt{h^2 + \omega_0^2}$ (Fig. 2e), and passing through the origin.

The third sheet corresponds to the phase plane satisfying the condition $(T_j\dot{\varphi} + K_j\varphi)\varphi \leq 0$ (Fig. 3c). The boundary of the sheet, corresponding to the equation $T_j\dot{\varphi} + K_j\varphi = 0$ is common to the second and third sheets.

* The method of multisheeted phase surfaces, as applied to automatic control systems, has been worked out in [18-20].

Thus, it is clear from an analysis of the phase planes constructed, that the optimal control system changes its structure three times during the transient response when the system operates in the stabilization mode, and twice when the system operates in the tracking mode. In both cases, the disturbing action is taken to be in the form of unit jumps.

The nonlinear control law which provides this shift of structure can be presented in the following form:

$$\xi(t) = \begin{cases} K\varphi + T\dot{\varphi} & \text{for} \quad \varphi\dot{\varphi} \geqslant 0, \\ K\varphi - T\dot{\varphi} & \text{for} \quad (T_j\dot{\varphi} + K_j\varphi)\dot{\varphi} < 0, \\ -K\varphi + T\dot{\varphi} & \text{for} \quad (T_j\dot{\varphi} + K_j\varphi)\varphi \leqslant 0. \end{cases} \qquad (14)$$

given by (14). It follows from expression (14) that the nonlinear link must contain a switching device which varies the signs of the error signal and its derivative during the transient response as a function of their signs and relationships.

For the realization of this control law, we us a nonlinear "key"-type correcting device consisting of two monotypic Ψ-cells [4] (Fig. 4).

The first Ψ_1-cell passes the signal $\dot{\varphi}$ only under the condition $(T_j\dot{\varphi} + K_j\varphi)\dot{\varphi} \leq 0$, and the second Ψ_2-cell passes signal φ under the condition that $(T_j\dot{\varphi} + K_j\varphi)\varphi \leq 0$. The static characteristics of these nonlinear devices are shown in Fig.5.†

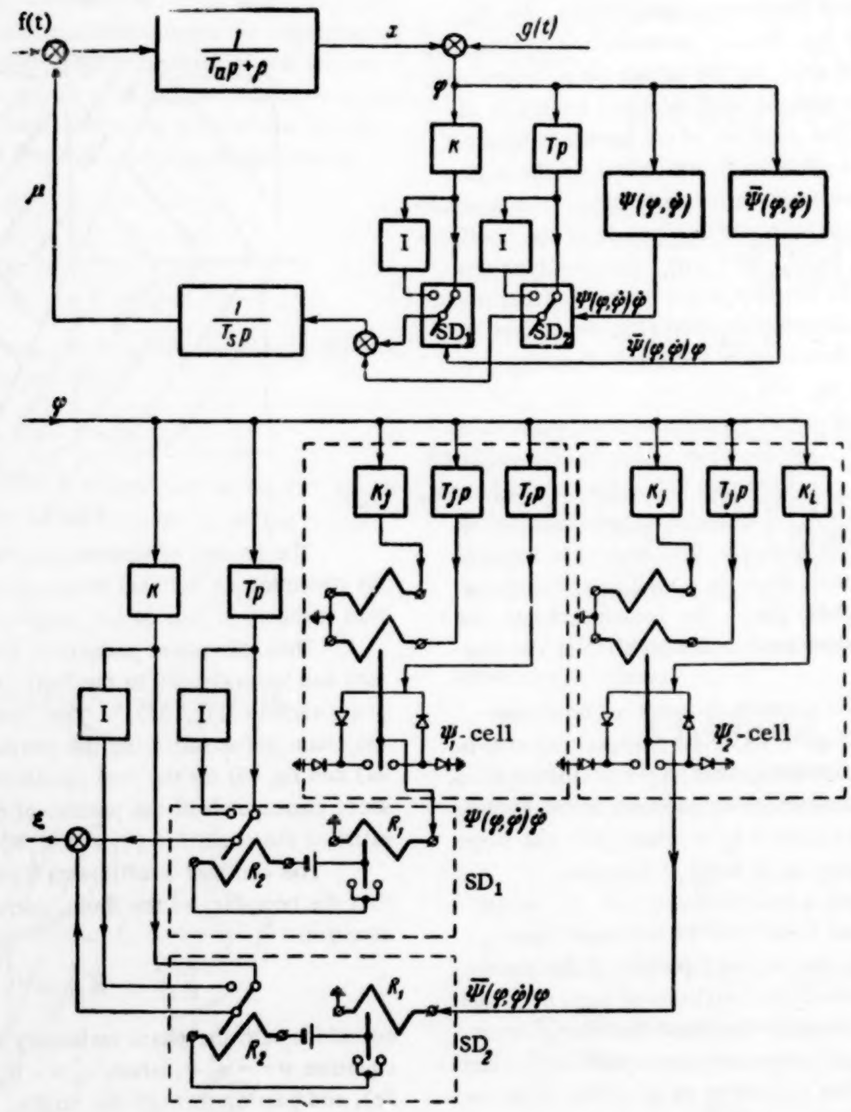

Fig. 4.

2. Realization of the Optimal Control Law

For the automatic control system to have the phase portrait shown in Fig. 3, it must contain a nonlinear correcting device which realizes the optimal control law

The signal from the output of these Ψ-cells acts on switching devices SD_1 and SD_2, consisting of the relay

† The nonlinear coefficients $\Psi(\varphi, \dot{\varphi})$ and $\overline{\Psi}(\varphi, \dot{\varphi})$ in the hatched portions of the $\varphi\dot{\varphi}$ plane equal the constant coefficients T_i and K_i, respectively. In the nonhatched portions they equal zero.

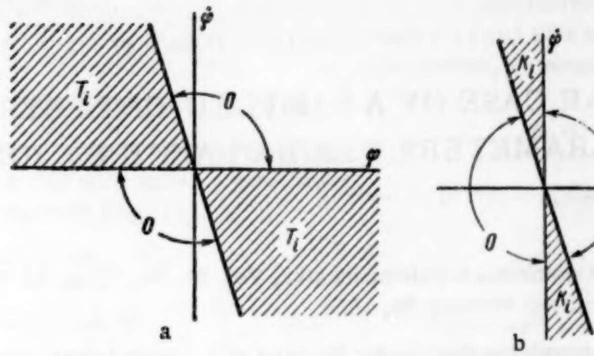

Fig. 5.

set R_1 and R_2. These devices provide the necessary changes in the signs of the error and its derivative by means of the inverter I.

The coefficients K_j and T_j of the correcting device are chosen on the basis of the relationship

$$\frac{K_j}{T_j} = h + \sqrt{h^2 + \omega_0^2}.$$

The connection between these coefficients and the system's parameters is defined by the following equation

$$\frac{K_j}{T_j} = \frac{T_s p + T}{2T_a T_s} + \sqrt{\frac{(T_s p + T)^2 + 4KT_a T_s}{4T_a^2 T_s^2}}. \quad (15)$$

Generally speaking, coefficients T_j and K_j can be chosen arbitrarily as functions of the dead zones of relays R_1 and R_2.

LITERATURE CITED

[1] V. V. Petrov, Investigation of a Pneumohydraulic Autopilot Servomechanism [in Russian] Dissertation (IAT AN SSSR, 1948).

[2] G. Lewis, "The use of nonlinear feedback for improving transient responses in automatic control systems," [Russian translation] Voprosy Raketnoi Tekhniki 4 (1954).

[3] G. M. Ostrovskii, "The use of nonlinear correcting devices in second-order automatic control systems," Avtomat. i Telemekh. 17, 11 (1956).‡

[4] V. A. Maslennikov, "Qualitative control of a neutral object by an astatic controller," Avtomat. i Telemekh. 17, 2 (1956).‡

[5] S. V. Emel'yanov, "A method for obtaining complex control laws using only the error signal or the controlled coordinate and its first derivative," Avtomat. i Telemekh. 18, 10 (1957).‡

[6] Automatic Control Fundamentals, [in Russian] ed. V. V. Solodovnikov (Mashgiz, 1954).

[7] I. Flügge-Lotz and C. F. Taylor, "Synthesis of a nonlinear control system," Trans. IRE, PG AC-1 (May, 1956).

[8] I. Flügge-Lotz and W. S. Wunch, "On a nonlinear transfer system," Journ. Appl. Phys. 26 (April, 1955).

[9] J. M. L. Janssen and J. C. Vermeulen, Behandlung zweier Regalungsprobleme mit Hysteresis und Reibung nach der Methode der Beschreibungsfunktion Nichtlineare Regelungs Forgange, Vortrage, (Munich, 1955).

[10] A. M. Letov, "Conditionally stable controlled systems (on one class of optimal controlled system)," Avtomat. i Telemekh. 18, 7 (1957).‡

[11] N. N. Bogolyubov and Yu. A. Mitropol'skii, Asymptotic Methods in Nonlinear Oscillation Theory [in Russian] (Mashgiz, 1955).

[12] A. A. Fel'dbaum, "The simplest relay systems of automatic control," Avtomat. i Telemekh. 10, 4 (1949).

[13] A. A. Fel'dbaum, "Optimal processes in automatic control systems," Avtomat. i Telemekh. 14, 6 (1953).

[14] A. A. Fel'dbaum, "On synthesizing optimal systems by means of phase space," Avtomat. i Telemekh. 16, 2 (1955).

[15] A. Ya. Lerner, "Improving the dynamic properties of automatic compensators by means of nonlinear connections, I and II," Avtomat. i Telemekh. 13, 2 and 4 (1952).

[16] A. Ya. Lerner, "On the limiting speed of response of automatic control systems," Avtomat. i Telemekh. 15, 6 (1954).

[17] V. V. Petrov and V. Yu. Rutkovskii, " Some questions in the construction of circuits and the choice of characteristics of fast-acting servomechanisms," in: Automatic Control and Computing Technology [in Russian], collected papers edited by V. V. Solodovnikov (Mashgiz, 1959) 2nd ed.

[18] V. V. Petrov and G. M. Ulanov, "Community of the skipping, vibration, and optimal modes of the class of servomechanisms," [in Russian]. Doklady Akad. Nauk SSSR 112, 3 (1957).

[19] V. V. Petrov and G. M. Ulanov, "Theory of the two simplest control systems," Avtomat. i Telemekh. 11, 5 (1950).

[20] V. V. Petrov, Mnoholistove a nektere trojrozmerne i lohy dynamiky nelinearnich servomechanismu souhrn praci o aut omatisaci, (Prague, 1957).

‡ See English translation.

ON ONE PARTICULAR CASE OF A SAMPLED-DATA SYSTEM WITH VARIABLE PARAMETERS WHICH CHANGE BY JUMPS

V. I. Teverovskii

(Dnepropetrovsk)

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A method is given for determining the transfer function of a sampled-data system with periodically varying parameters in the particular case when the system is comprised of a series connection of a first-order link with variable parameters and links with constant parameters.

Among systems with variable parameters, a special place is occupied by systems whose parameters vary periodically at the pulse repetition frequency, remaining constant during the pulses and in the intervals between pulses, and changing by jumps at the boundaries of these time segments. Works by F. M. Kilin, E. A. Rozenman, and others have been devoted to these systems. A method for designing such systems in the general case was given in [1]. This method leads to cumbersome computations. In [2] Ya. Z. Tsypkin presented a simple method for investigating first-order systems.

The attempt is made in the present work to design sampled-data systems with periodically varying parameters for the very important particular case when only one root of the characteristic equation changes. As will be shown, this means physically that the system consists of a series connection of a first-order link with variable parameters and a system with constant parameters.

1. The First-Order Link*

We consider a link described by the equation

$$(q-q_0)\,x(\bar{t}) = -\,kq_0 f(\bar{t}). \tag{1.1}$$

Here and in the sequel we shall use the dimensionless variables $\bar{t} = t/T$, $q = pT$ ($p = d/dt$), where T is the repetition period of the input pulses.

The parameters q_0 and k are variable, and are subject to the following relationships:

$$\begin{aligned} q_0 &= q_1 \\ k &= k_1 \end{aligned} \quad \text{for} \quad n \leqslant t \leqslant n+\gamma,$$

$$\begin{aligned} q_0 &= q_1' \\ k &= k_1' \end{aligned} \quad \text{for} \quad n+\gamma < \bar{t} < n+1.$$

Here, $n = 0, 1, 2, \ldots$, $\gamma = \tau/T$, τ is the duration of the input pulses,

$$f(\bar{t}) = f[n] \quad \text{for} \quad n \leqslant \bar{t} \leqslant n+\gamma$$

and

$$f(\bar{t}) = 0 \quad \text{for} \quad n+\gamma < \bar{t} < n+1.$$

By solving Eq. (1.1) for the interval $(n, n+\gamma)$ with initial condition $x[n]$ and for the interval $(n+\gamma, n+1)$ with initial condition $x[n+\gamma]$, and by also taking into account that no jump in the initial conditions occurs at the boundary of the intervals cited, we obtain the difference equation

$$x[n+1] = \\ = k_1 f[n]\,(1-e^{q_1\gamma})\,e^{q_1'(1-\gamma)} + x[n]\,e^{q_1\gamma + q_1'(1-\gamma)}. \tag{1.2}$$

If we take the discrete Laplace transform of Eq.(1.2), taking into account the lead theorem and the condition x(0) = 0 [3], we obtain

$$X^*(q) = k_1\,\frac{(1-e^{q_1\gamma})\,e^{q_1'(1-\gamma)}}{e^{q} - e^{q_{1\,\mathrm{av}}}}\,F^*(q), \tag{1.3}$$

where $X^*(q)$ and $F^*(q)$ are the discrete Laplace transforms of the step functions x[n] and f[n], respectively, and

$$q_{1\,\mathrm{av}} = q_1\gamma + q_1'(1-\gamma).$$

Consequently, the transfer function of the first-order sampled-data system with variable parameters has the form:

$$W^*(q) = k_1\,\frac{(1-e^{q_1\gamma})e^{q_1'(1-\gamma)}}{e^{q} - e^{q_{1\,\mathrm{av}}}}. \tag{1.4}$$

We now determine the impulsive response of system (1.4) at an arbitrary moment of time.

Letting $\bar{t} - n = \varepsilon$, we get

$$w[n,\varepsilon] = k_1\,(e^{-q_1\gamma}-1)\,e^{q_{1\,\mathrm{av}}n}e^{q_1\varepsilon} \qquad (0 \leqslant \varepsilon \leqslant \gamma,\ \ n>0), \tag{1.5}$$

$$w[n,\varepsilon] = k_1\,(1-e^{q_1\gamma})\,e^{q_{1\,\mathrm{av}}n}e^{q_1'(\varepsilon-\gamma)} \qquad (\gamma < \varepsilon < 1, n > 0), \tag{1.6}$$

$$w[0,\varepsilon] = k_1\,(1-e^{q_1\varepsilon}) \qquad (0 \leqslant \varepsilon \leqslant \gamma). \tag{1.7}$$

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* The investigation of such links was provided in [2].

$$w[0, \varepsilon] = k_1 (1 - e^{q_1 \gamma}) e^{q_1' (\varepsilon - \gamma)} \qquad (\gamma < \varepsilon < 1). \tag{1.8}$$

2. The Second-Order System

We now consider the second-order system consisting of a series connection of the link with variable parameters and a link with constant parameters (Fig. 1):

$$(q - q_{10}) x_1 (\bar{t}) = - k q_{10} f(\bar{t}),$$

$$(q - q_2) x(\bar{t}) = - q_2 x_1 (\bar{t}). \tag{2.1}$$

Here, $q_2 = \mathrm{const}$, $q_{10} = q_1$, and $k = k_1$ when $n \le \bar{t} \le n + \gamma$, $q_{10} = q'_1$ and $K = k'_1$ when $n + \gamma < \bar{t} < n + 1$.

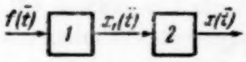

Fig. 1.

We now determine the impulsive response of the system described by Eqs. (2.1).

In this case, the input to link 2 with constant parameters is signal $x_1(\bar{t})$, the impulsive response of link 1 with variable parameters [cf. Eqs. (1.5)-(1.8)]:

$$(q - q_2) w_2 [n, \varepsilon] = - q_2 w_1 [n, \varepsilon]. \tag{2.2}$$

The solution is carried out for the time interval (n, n + 1).

The result of solving Eq. (2.2) with initial condition $w_2[n]$ is

$$w_2 [n + \gamma] = k_1 (e^{-q_1 \gamma} - 1) \frac{q_2}{q_1 - q_2} (e^{q_2 \gamma} - e^{q_1 \gamma}) \times$$

$$\times e^{q_{1\mathrm{av}} n} + w_2 [n] e^{q_2 \gamma}.$$

By now solving Eq. (2.2) with initial condition $w_2(n + \gamma)$ and then carrying out some elementary transformations, we obtain the following difference equation:

$$w_2\ n + 1] = w_2 [n] e^{q_2} -$$

$$- k_1 (e^{-q_1 \gamma} - 1) \frac{q_2}{q_1 - q_2} (e^{q_2(1-\gamma) + q_1 \gamma} - e^{q_2}) e^{q_{1\mathrm{av}} n} -$$

$$- k_1 (1 - e^{q_1 \gamma}) (e^{q_1' (1-\gamma)} - e^{q_2(1-\gamma)}) \frac{q_2}{q_1' - q_2} e^{q_{1\mathrm{av}} n}. \tag{2.3}$$

As is well known, one gives the name operator, or transfer function, of a sampled-data system to a series of the form

$$W^*(q) = \sum_{n=1}^{\infty} w[n] e^{-qn} \quad \text{for} \quad w(0) = 0, \tag{2.4}$$

where w[n] is the impulsive response of the corresponding continuous system.

It may be easily shown that

$$\sum_{n=1}^{\infty} w[n + 1] e^{-qn} = e^{q} \sum_{n=1}^{\infty} w[n] e^{-qn} - w[1]. \tag{2.5}$$

Indeed, by making the substitution n = k - 1, we get

$$\sum_{n=1}^{\infty} w[n + 1] e^{-qn} = e^{q} \sum_{k=2}^{\infty} w[k] e^{-qk}.$$

By adding and substracting w[1] in the right member, we obtain (2.5).

If we multiply both sides of Eq. (2.3) by e^{-qn} and then sum them over n from 1 to infinity, taking into account (2.5) and the equality [3], we get

$$\sum_{n=1}^{\infty} e^{\alpha n} e^{-qn} = \frac{e^{\alpha}}{e^{q} - e^{\alpha}},$$

$$W_2^*(q) = k_1 \frac{\left[(1 - e^{-q_1 \gamma}) (e^{q_2(1-\gamma) + q_1 \gamma} - e^{q_2}) \frac{q_2}{q_1 - q_2} + (e^{q_1 \gamma} - 1) \frac{q_2}{q_1' - q_2} (e^{q_1' (1-\gamma)} - e^{q_2(1-\gamma)}) \right] e^{q_{1\mathrm{av}}}}{(e^{q} - e^{q_2}) (e^{q} - e^{q_{1\mathrm{av}}})} + \frac{w_2[1]}{e^{q} - e^{q_2}}; \tag{2.6}$$

$w_2[1]$ is determined analogously to $w_2[n + 1]$ on the basis of expressions (1.7) and (1.8), account being taken of the fact that $w_2(0) = 0$.

As the result, we obtain

$$w_2[1] = k_1 \frac{q_2 (q_1 - q_2) e^{q_{1\mathrm{av}}} - q_1 (q_1' - q_2) e^{q_2} + q_2 (q_1' - q_1) e^{q_1 \gamma + q_2(1-\gamma)}}{(q_1 - q_2)(q_1' - q_2)} + \frac{(q_2 - q_1)(q_2 e^{q_1'(1-\gamma)} - q_1' e^{q_2(1-\gamma)})}{(q_1 - q_2)(q_1' - q_2)}. \tag{2.7}$$

By taking (2.7) into account, we obtain the final expression for $W_2^*(q)$:

$$W_2^*(q) = k_1 \frac{b_1 e^{q} + b_0}{(e^{q} - e^{q_2})(e^{q} - e^{q_{1\mathrm{av}}})}, \tag{2.8}$$

where

$$b_0 = e^{q_2 + q_{1\mathrm{av}}} + \frac{q_2 e^{q_2 + q_1' (1-\gamma)} - q_1 e^{q_{1\mathrm{av}} + q_2(1-\gamma)}}{q_1 - q_2}, \tag{2.9}$$

$$b_1 = \frac{w_2[1]}{k_1}. \tag{2.10}$$

If the system is astatic and its linear portion has a transfer function of the form

$$W_a(q) = \frac{-kq_{10}}{(q-q_{10})\,q}, \tag{2.11}$$

it is then easily seen that the corresponding discrete transfer function can be obtained by passing to the limit:

$$W_a^*(q) = \lim_{q_2 \to 0}\left[-\frac{1}{q_2} W_2^*(q)\right]. \tag{2.12}$$

By carrying out some elementary transformations, we get

$$W_a^*(q) = k_1 \frac{[(q_1' - q_1)(1 - e^{q_1\gamma}) + q_1(e^{q_1'(1-\gamma)} - e^{q_1 \mathrm{av}}) + \gamma q_1 q_1']\, e^q}{q_1 q_1' (e^q - e^{q_1 \mathrm{av}})(e^q - 1)} + \frac{q_1'(e^{q_1 \mathrm{av}} - e^{q_1'(1-\gamma)} - \gamma q_1 e^{q_1 \mathrm{av}})}{q_1 q_1' (e^q - e^{q_1 \mathrm{av}})(e^q - 1)}. \tag{2.13}$$

In the very important particular case when $\gamma \ll 1$, the expression for $W^*(q)$ is significantly simplifed. By taking into account that $e^{q_i\gamma} \approx 1 + q_i\gamma$ (if $|q_i\gamma| \ll 1$), we get

$$W_2^*(q) = k_1 \frac{\gamma q_1 q_2 (e^{q_1} - e^{q_2}) e^q}{(q_1' - q_2)(e^q - e^{q_1 \mathrm{av}})(e^q - e^{q_2})}. \tag{2.14}$$

3. A System of Arbitrary Order with One Variable Root of the Characteristic Equation

Let there be given a system of arbitrary order r. One of the roots of the system's characteristic equation is variable, all the roots being assumed to be simple. It may then be shown that the system takes the form of a series connection of a variable link 1 and a constant portion 2 which is described by an equation of order (r-1).

Fig. 2.

The transfer function of link 1 (Fig. 2) is

$$W_1(q) = -k_1 \frac{q_{10}}{q - q_{10}}, \tag{3.1}$$

where q_{10} is the variable root, assuming the values of q_1 and q_1'.

The transfer function of the constant portion 2 is

$$W_{r-1}(q) = \frac{P(q)}{Q(q)}, \tag{3.2}$$

where P(q) and Q(q) and polynomials in q, the order of P(q) being less than the order of Q(q).

Then,

$$W_{r-1}(q) = \frac{P(q)}{Q(q)} = \sum_{k=1}^{r-1} \frac{P(q_k)}{Q'(q_k)} \frac{1}{q - q_k} = \sum_{k=1}^{r-1} \frac{c_k}{q - q_k}, \tag{3.3}$$

where q_k are the roots of the equation $Q(q) = 0$, $Q'(q_k) = dQ(q)/dq\,|_{q = q_k}$.

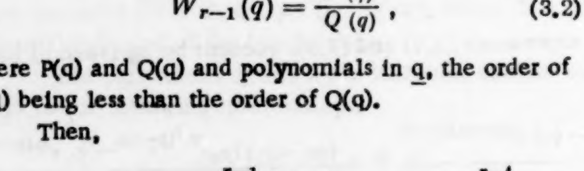

Fig. 3.

On the basis of (3.3), the block diagram of the system may be given as shown on Fig. 3 a.†

On Fig. 3 a, $q_1, q_2, \ldots, q_{r-1}$ denote the links whose transfer functions are the terms of sum (3.3); Σ is a summing device.

Obviously, the processes in the system are not changed if the block schematic is varied so as to take the form shown in Fig. 3 b. But, it is clear from Fig. 3 b, that an rth-order system with variable parameters can be given in the form of a parallel connection of r-1 second-order systems and, consequently, the discrete transfer function of such a system has the form:

$$W^*(q) = \sum_{k=1}^{r-1} c_k W_k^*(q), \tag{3.4}$$

where

$$W_k^*(q) = -\frac{1}{q_k} \frac{b_{1k} e^q + b_{0k}}{(e^q - e^{q_1 \mathrm{av}})(e^q - e^{q_k})},$$

$$b_{1k} = \frac{q_1(q_k - q_1') e^{q_k} + q_k(q_1' - q_1) e^{q_k(1-\gamma) + q_1\gamma}}{(q_1 - q_k)(q_1' - q_k)} +$$

$$+ \frac{q_k(q_1 - q_k) e^{q_1 \mathrm{av}} + (q_1 - q_k)(q_1' e^{q_k(1-\gamma)} - q_k e^{q_1'(1-\gamma)})}{(q_1 - q_k)(q_1' - q_k)}$$

$$b_{0k} = e^{q_1 \mathrm{av} + q_k} + \frac{q_k e^{q_k + q_1'(1-\gamma)} - q_1 e^{q_1 \mathrm{av} + q_k(1-\gamma)}}{q_1 - q_k}. \tag{3.5}$$

If $q_k = 0$, then expression (2.13) can be used for computing $W_k^*(q)$.

If $\gamma \ll 1$, then $W_k^*(q)$ can be computed by formula (2.14) by replacing q_2 by q_k in its right member and dividing by $(-q_k)$.

† A similar representation of the schematic is given in [1].

Up till now, we have assumed that the first link is a simple inertial one. If, however, the transfer function of this link has the form:

$$W_1(q) = k_1 \frac{q+\alpha}{q+\beta}, \tag{3.6}$$

where k_1, α, and β are variable parameters, then the link may be presented in the form of a parallel connection of an inertial link and an ideal amplifier with variable parameters:

$$W_1(q) = k_1 - k_1 \frac{\beta - \alpha}{q+\beta}. \tag{3.7}$$

Since the input to the variable system is $f(\bar{t})$ at times $n + \gamma < \bar{t} < n + 1$, varying the amplifier parameters does not affect the process, and the discrete transfer function of the second-order system whose first link has a transfer function of the form (3.7) is obtained in the form of a difference of the discrete first-order transfer function corresponding to the second link, but multiplied by k_1, and the discrete transfer function of the second-order system whose first link has a transfer function of the form $k_1(\beta - \alpha)/(q+\beta)$.

Thus, for a sampled-data system with periodically varying parameters, it is sufficiently simple to determine the discrete transfer function which has constant parameters (in the case when only one root of the characteristic equation varies). It is particularly simple to obtain the expression when the pulses are of short duration ($\gamma \ll 1$).

In conclusion, we consider an example of the application of the method presented above.

4. Transient Response in a Radar Range Finder

The range finder's operation is based on the comparison of the position of the pulse reflected from the target with the position of a pair of special selector pulses – semistrobe S and the translation of S after the translation in time of the echo pulse EP.

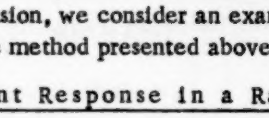

Fig. 4.

The block schematic of the range-finder is given in Fig. 4.

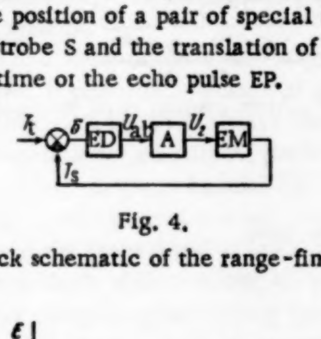

Fig. 5.

The scheme shown does not, of course, represent the contemporary state of radar technology, but is very convenient as an example of the application of the method presented here.

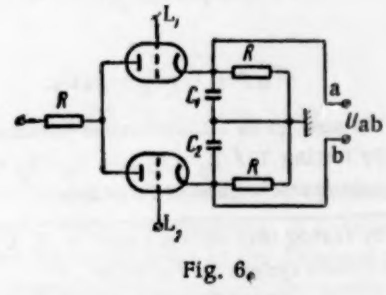

Fig. 6.

The nomenclature used on Fig. 4 is as follows: ED is the error discriminator, A is an amplifier, EM is the executive mechanism, T_t is the temporal position of the middle of the echo pulse from the target with respect to the sonde pulse SP, and T_s is the position of the semistrobe boundary (Fig. 5). The ED processes the voltage proportional to the quantity $\delta = T_t - T_s$. We now consider one discriminator circuit (Fig. 6).

Tubes L_1 and L_2 are normally cut off, and are opened by the semistrobes. If the center of the EP coincides with the boundary of a semistrobe, the charges taken by capacitors C_1 and C_2 during a period are equal, and $U_{ab} = 0$.

If the EP is displaced relative to the semistrobes, the charges taken by C_1 and C_2 are unequal, and $U_{ab} \neq 0$.

It is easily seen that the error discriminator parameters are different when the semistrobes are present from what they are when the semistrobes are absent.

Strictly speaking, the system under consideration is a system with pulse-width modulation (PWM). But with low porosity $\gamma = \tau / 2T_r$ (τ is the total length of the semistrobes and T_r is the repetition period of the sonding pulses) and if the condition $\tau/2 \ll T_c$ holds (T_c is the time constant of the charging of capacitors $C_1 = C_2 = C$), one can replace the system with PWM by a system with pulse-amplitude modulation with a constant pulse duration equal to $\tau/2$. Moreover, the time for switching the discriminator parameters is defined by the duration of the semistrobes $\tau/2 = \text{const}$.

From what has been said above, we obtain the following equations for the discriminator.

$$(T_c p + 1) U_{ab} = k_1 \frac{R}{R+R_1} \delta \quad \text{for} \quad nT_r < t < nT_r + \frac{\tau}{2} \tag{4.1}$$

and

$$(T_p p + 1) U_{ab} = 0 \quad \text{for} \quad nT_r + \frac{\tau}{2} < t < (n+1) T_r \tag{4.2}$$

Here, $k_1 = 4E/\tau$, E is the amplitude of the echo pulse at the discriminator's input,

$$T_c = \frac{CRR_1}{R+R_1} \approx CR_1, \; T_p = RC.$$

The amplifier and the executive mechanism are links with constant parameters. Their equations are, respectively,

$$U_2 = k_2 U_{\text{ab}}, \tag{4.3}$$

$$(T_M p + 1)\, p T_s = k_3 U_2. \tag{4.4}$$

We now go to dimensionless parameters.

By letting $T_r / T_p = a_1$ and $T_c / T_p = \alpha$, we obtain the discriminator's transfer function:

$$W_1(q) = k_1 \frac{(1-\alpha')\,\dfrac{a_1}{\alpha'}}{q + \dfrac{a_1}{\alpha'}}, \tag{4.5}$$

where $\alpha' = \alpha$ for $n \le \bar{t} \le n + \gamma$ and $\alpha' = 1$ for $n = \gamma < \bar{t} < n + 1$.

The transfer function of the amplifier and the EM takes the form

$$W_2(q) = \frac{k_2 k_3 T_r \, a_2}{(q + a_2)\, q} = k_2 k_3 T_r \left(\frac{1}{q} - \frac{1}{q + a_2}\right). \tag{4.6}$$

By taking into account that $\gamma \ll 1$ and using formula (2.14), we obtain the transfer function of the open-loop sampled-data system in the form

$$W^*(q) = k_1 k_2 k_3 T_r \,(1-\alpha)\, \frac{\gamma}{\alpha} \left[\frac{(1 - e^{-a_1})\, e^q}{(e^q - e^{-a_{1\,\text{av}}})(e^q - 1)} - \frac{a_1 (e^{-a_1} - e^{-a_2})\, e^q}{(a_2 - a_1)(e^q - e^{-a_2})(e^q - e^{-a_{1\,\text{av}}})}\right].$$

where

$$a_{1\,\text{av}} = \frac{a_1}{\alpha}\gamma + a_1(1-\gamma) \approx a_1\left(1 + \frac{\gamma}{\alpha}\right).$$

After some elementary transformations, we finally obtain

$$W^*(q) = k \frac{\left(1 - \dfrac{a_1 e^{-a_2} - a_2 e^{-a_1}}{a_1 - a_2}\right) e^{2q} + \left(e^{-(a_1+a_2)} - \dfrac{a_1 e^{-a_1} - a_2 e^{-a_2}}{a_1 - a_2}\right) e^q}{(e^q - e^{-a_{1\,\text{av}}})(e^q - e^{-a_2})(e^q - 1)}. \tag{4.7}$$

If the transfer function of the open-loop sampled-data system has the form:

$$W^*(q) \doteq k \frac{B_2 e^{2q} + B_1 e^q}{A_3 e^{3q} - A_2 e^{2q} + A_1 e^q - A_0},$$

then, starting from the analogy of the Nyquist criterion, one can show that the following inequality must hold in order that the closed-loop system be stable:

$$k < \left|\frac{A_0^2 + A_1 A_3 - A_0 A_2 - A_3^2}{A_0 B_2 + A_3 B_1}\right|. \tag{4.8}$$

For the given case, inequality (4.8) takes the form

$$k < \frac{(1 - e^{-(a_2 + a_{1\,\text{av}})})(e^{-a_2} + e^{-a_{1\,\text{av}}} - e^{-(a_2 + a_{1\,\text{av}})})}{e^{-(a_2 + a_{1\,\text{av}})}\left(1 - \dfrac{a_1 e^{-a_2} - a_2 e^{-a_1}}{a_1 - a_2}\right) + e^{-(a_1+a_2)} - \dfrac{a_1 e^{-a_1} - a_2 e^{-a_2}}{a_1 - a_2}}. \tag{4.9}$$

If $a_2 \ll 1$ and $a_1 \ll 1$, which corresponds to $T_p \gg T_r$ and $T_M \gg T_r$, then inequality (4.9) simplifies to

$$k < \frac{(1 - e^{-a_{1\,\text{av}}})(1 - e^{-a_{1\,\text{av}}} + a_2 e^{-a_{1\,\text{av}}})}{a_1}. \tag{4.10}$$

If the condition $a_{1\,\text{av}} \ll 1$ also holds, then condition (4.9) takes the form:

$$k < \frac{a_2 a_{1\,\text{av}} + a_{1\,\text{av}}^2}{a_1}. \tag{4.11}$$

In conclusion, the author wishes to express his gratitude to Ya. Z. Tsypkin for his aid in the preparation of this paper.

LITERATURE CITED

[1] F. M. Kilin, "Transient responses and steady-state processes in sampled-data systems with variable parameters which change by jumps," Avtomat. i Telemekh. 18, 12 (1957).‡

[2] Ya. Z. Tsypkin, Theory of Sampled-Data Systems [in Russian] (Fizmatgiz, 1958).

[3] Ya. Z. Tsypkin, Transient Responses and Steady-State Processes in Sampled-Data Circuits [in Russian] (Gosénergoizdat, 1951).

[4] Ya. Z. Tsypkin, "Theory of discontinuous control, I, II, "Avtomat. i Telemekh. 10, 3 and 5 (1949).

[5] F. M. Kilin, "Transient responses and steady-state processes in automatic range-finders," Avtomat. i Telemekh. 19, 10 (1958).‡

‡ See English translation.

ON METHODS OF SIMULATING RATIONAL-FRACTION FUNCTIONS WITHOUT THE USE OF DIFFERENTIATING ELEMENTS

B. Ya. Kogan

(Moscow)

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Methods of simulating rational-fraction transfer functions without the use of differentiating elements when the original differential equations contain constant and variable coefficients are compared. It is asserted that the minimum number of operational amplifiers required is n + 3, where n is the order of the differential equation. For simplicity of the circuit and for the minimum amount of preliminary calculation, the method of combining derivatives is preferable for constant coefficients, whereas the method of going over to the equivalent system of first-order equations is preferable when the coefficients are variable.

In the investigation of automatic control and regulation systems by means of electronic analog computers, the necessity frequently arises of reproducing rational-fraction transfer functions of the form

$$W(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}, \quad (1)$$

where

$$m \leqslant n,$$

and $b_m, b_{m-1}, \ldots, b_1, b_0$ and $a_{n-1}, a_{n-2}, \ldots, a_1, a_0$ are given constant coefficients.

Transfer functions of this type are obtained, for example, from the approximate representation of lags by a Padé series, the synthesis of correcting circuits, and a number of problems in the statistical dynamics of automatic control systems.

The use of differentiators in the setups for these transfer functions is proscribed because of the sharp amplification of the noise which is always found in the output signals of operational amplifiers, the representation of the right members as given functions of time being possible only in a very limited number of cases when the law of change of the input signal is known beforehand.

Various methods of reproducing these transfer functions using only integrating and adding elements are known from the literature [1-4]. These methods can be reduced to four basic ones:

1) direct integration;
2) decomposition of the transfer function into simpler ones (method of structure transformation);
3) decomposition into first-order equations;
4) combination of derivatives.

In the literature, these methods were not compared among themselves, and some of them were mentioned only superficially. It is also of interest to establish the possibility of extending them to the case when the coefficients of the differential equation giving rise to (1) are given functions of time.

1. Method of Direct Integration

It is convenient to consider this method in terms of the simple example of the equation

$$a_3y''' + a_2y'' + a_1y' + a_0y =$$
$$= b_0x + b_1x' + b_2x'' + b_3x'''$$

with given constant coefficients a_0, a_1, a_2, and a_3, and b_0, b_1, b_2, and b_3, initial conditions $y(0)$, $y'(0)$, and $y''(0)$, and a disturbance x whose dependence on time is not given beforehand.

We solve the initial equation for the difference of the highest derivatives:

$$a_3y''' - b_3x''' =$$
$$= -(a_2y'' - b_2x'') - (a_1y' - b_1x') - (a_0y - b_0x).$$

By introducing the symbolic integration operator $\int dt = 1/p$, $\int\int dt\, dt = 1/p^2$, etc., we obtain

$$a_3y - b_3x = -\frac{1}{p}(a_2y - b_2x) -$$
$$- \frac{1}{p^2}(a_1y - b_1x) - \frac{1}{p^3}(a_0y - b_0x). \quad (2)$$

In order to retain the initial values of the coefficients a_0, a_1, a_2, a_3, b_0, b_1, b_2, and b_3 in the setup of the problem, we add the term $-(a_2-1)y$ to the right and left sides of expression (2). The result is

$$y = -\frac{1}{p}(a_2y - b_2x) - \frac{1}{p^2}(a_1y - b_1x) -$$
$$- \frac{1}{p^3}(a_0y - b_0x) + b_3x - (a_3 - 1)\,y. \quad (3)$$

Expression (3) allows the structural scheme (block schematic) of the setup to be given directly. Indeed, if we denote by the new variable z_1 the sum of all the terms

in (3) which contain the symbol for integration, we obtain the equation

$$y = z_1 + b_3 x - (a_3 - 1)\, y. \tag{4}$$

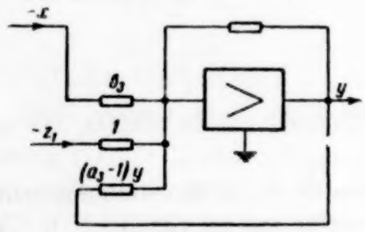

Fig. 1.

The quantity y, in correspondence with (4), can be formed by means of an adder, as shown on Fig. 1. It should be mentioned that such a method for isolating y gives a stable setup only in case $a_3 > 1$. If $a_3 < 1$, it is necessary to divide all the coefficients by a_3 as a preliminary step. With this, there is no longer the necessity of introducing the term $(a_3-1)y$ and, in the problem setup, there appear the new values of the coefficients:

$$\frac{a_2}{a_3},\quad \frac{a_1}{a_3},\quad \frac{a_0}{a_3};\quad \frac{b_0}{a_3},\quad \frac{b_1}{a_3},\quad \frac{b_2}{a_3},\quad \frac{b_3}{a_3}.$$

For the further construction of the scheme, we determine the value of the product pz_1:

$$pz_1 = -(a_2 y - b_2 x) - \frac{1}{p}(a_1 y - b_1 x) - \frac{1}{p^2}(a_0 y - b_0 x).$$

By introducing the notation

$$z_2 = -\frac{1}{p}(a_1 y - b_1 x) - \frac{1}{p^2}(a_0 y - b_0 x),$$

we get

$$pz_1 = -(a_2 y - b_2 x) + z_2. \tag{5}$$

Consequently, it is necessary to add an integrator to the schematic of Fig. 1. By continuing similarly, we arrive at the expression

$$pz_2 = -(a_1 y - b_1 x) + z_3, \tag{6}$$

where

$$pz_3 = -(a_0 y - b_0 x),\quad z_3 = -\frac{1}{p}(a_0 y - b_0 x).$$

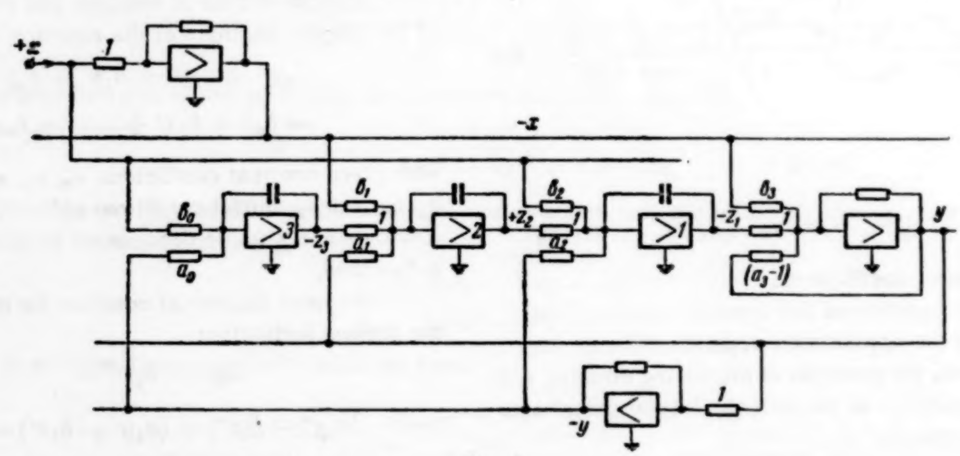

Fig. 2.

Equations (4), (5), and (6) lead directly to the setup shown on Fig. 2. For $m = n$, the total number of blocks required here is $n + 3$. For $m < n$, one can obviously dispense with the output adder, so that the number of blocks will be $n + 2$.

The initial conditions for integrators 1, 2, and 3 can be computed by giving the relationships for z_1, z_2, and z_3 in the form

$$z_1(0) = a_3 y(0) - b_3 x(0),$$
$$z_2(0) = a_3 y'(0) - b_3 x'(0) + a_2 y(0) - b_2 x(0),$$
$$z_3(0) = a_3 y''(0) - b_3 x''(0) + a_2 y'(0) - b_2 x'(0) + a_1 y(0) - b_1 x(0).$$

The regularities apparent in the relationships given allow one to write directly the relationships for the initial conditions for the kth integrator:

$$z_k(0) = z_{n-j} = \sum_{1}^{n-j} [a_{(i+j)} y^{(i-1)}(0) - b_{(i+j)} x^{(i-1)}(0)], \tag{7}$$

where $j = n - k$, $i = 1, 2, 3, \ldots$

The basic advantage of the method just considered is that the setup is implemented from the original coefficients, and the computation of the initial conditions is carried out with comparative simplicity.

2. Decomposition of the Transfer Function into Simpler Ones

The basis of this method is that any transfer function $W(s)$ can be considered as the transfer function of some single-loop system with negative feedback, in which the forward path has transfer function $W_1(s)$ and the feedback path has transfer function $W_2(s)$. The conditions imposed on the functions W_1 and W_2 are that the numerator of the first may not contain terms with s, and the numerator of the second may be a polynomial in s which is of degree one less than the polynomial in the numerator of $W(s)$.

Indeed, let

$$W(s) = \frac{R(s)}{P(s)} = \frac{W_1(s)}{1 + W_1(s)\, W_2(s)}. \tag{8}$$

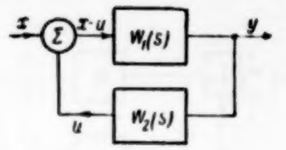

Fig. 3.

The block schematic of the connection of $W_1(s)$ and $W_2(s)$ is given in Fig. 3.

Let

$$W_1(s) = \frac{y(s)}{x(s) - u(s)} = \frac{1}{Q_1(s)},$$

$$W_2(s) = \frac{u(s)}{y(s)} = \frac{Q_3(s)}{Q_2(s)}. \qquad (9)$$

We now find the conditions which must be satisfied by $Q_1(s)$, $Q_2(s)$, and $Q_3(s)$ so that Eq. (8) may be valid:

$$\frac{R(s)}{P(s)} = \frac{Q_2(s)}{Q_1(s)\,Q_2(s) + Q_3(s)}, \qquad (10)$$

whence

$$R(s) = Q_2(s), \quad P(s) = Q_1(s)\,Q_2(s) + Q_3(s). \qquad (11)$$

It follows from conditions (11) that the degree of polynomial $Q_1(s)$ must be equal to the difference of the degrees of polynomials $P(s)$ and $R(s)$, and from (9) and (10) that the degree of polynomial $Q_3(s)$ must be one less than the degree of polynomial $R(s)$. By using these special characteristics, we can write the general form of the polynomials $Q_1(s)$ and $Q_3(s)$:

$$Q_1(s) = \sum_{i=0}^{n-m} l_i s^i, \quad Q_3(s) = \sum_{i=0}^{m-1} m_i s^i, \qquad (12)$$

where l_i and m_i are constants to be determined.

For example, let $R(s) = b_0 + b_1 s + b_2 s^2 + b_3 s^3$ and $P(s) = a_0 + a_1 s + a_2 s^2 + s^3$. Then, in accordance with (12), we obtain

$$Q_1(s) = l_0, \quad Q_2(s) = b_0 + b_1 s + b_2 s^2 + b_3 s^3,$$

$$Q_3(s) = m_0 + m_1 s + m_2 s^2.$$

To determine the values of l_0, m_0, m_1, and m_2, we substitute the values found for Q_1, Q_2, and Q_3 in (11). The result is

$$a_0 + a_1 s + a_2 s^2 + s^3 = l_0 b_3 s^3 + (l_0 b_2 + m_2) s^2 + (l_0 b_1 + m_1) s + l_0 b_0 + m_0,$$

whence the coefficients sought will be

$$l_0 = \frac{1}{b_3}, \quad m_0 = a_0 - \frac{b_0}{b_3},$$

$$m_1 = a_1 - \frac{b_1}{b_3}, \quad m_2 = a_2 - \frac{b_2}{b_3}.$$

Thus,

$$W_1(s) = \frac{1}{l_0}, \quad W_2(s) = \frac{m_0 + m_1 s + m_2 s^2}{b_0 + b_1 s + b_2 s^2 + b_3 s^3}.$$

In its turn, $W_2(s)$ can be presented as

$$W_2(s) = \frac{W_{11}(s)}{1 + W_{11}(s)\,W_{22}(s)},$$

where

$$W_{11}(s) = \frac{1}{n_0 + n_1 s}, \quad W_{22}(s) = \frac{c_0 + c_1 s}{m_0 + m_1 s + m_2 s^2}.$$

To determine the unknown coefficients we use, based on (11), the relationship

$$b_0 + b_1 s + b_2 s^2 + b_3 s^3 = (n_0 + n_1 s)(m_0 + m_1 s + m_2 s^2) + (c_0 + c_1 s).$$

By equating coefficients, we get

$$c_0 = b_0 - \left(b_2 - b_3 \frac{m_1}{m_2}\right) \frac{m_1}{m_2},$$

$$c_1 = b_1 - \left[b_3 \frac{m_0}{m_2} + \frac{m_1}{m_2}\left(b_2 - b_3 \frac{m_1}{m_2}\right)\right],$$

$$n_0 = \frac{b_2 - b_3 \frac{m_1}{m_2}}{m_2}, \quad n_1 = \frac{b_3}{m_2}.$$

We now decompose $W_{22}(s)$ into the simpler components:

$$W_{22}(s) = \frac{W_{111}(s)}{1 + W_{111}(s)\,W_{222}(s)},$$

$$W_{111}(s) = \frac{1}{q_0 + q_1 s}, \quad W_{222}(s) = \frac{r_0}{c_0 + c_1 s}.$$

We determine the coefficients r and q from the relationship

$$m_0 + m_1 s + m_2 s^2 = (q_0 + q_1 s)(c_0 + c_1 s) + r_0,$$

whence

$$q_1 = \frac{m_2}{c_1}, \quad q_0 = \left(m_1 - \frac{c_0}{c_1} m_2\right)\frac{1}{c_1},$$

$$r_0 = m_0 - \left(m_1 - m_2 \frac{c_0}{c_1}\right)\frac{c_0}{c_1}.$$

Thus, the problem leads to the setup of the following system of equations:

$$\begin{aligned} y &= \frac{1}{l_0}(x - u), \\ (n_0 + n_1 s)\,u &= y - u_1, \\ (q_0 + q_1 s)\,u_1 &= u - u_2, \\ (c_0 + c_1 s)\,u_2 &= r_0 u_1. \end{aligned} \qquad (13)$$

The block schematic and the setup which correspond to this system of equations are given in Figs. 4 and 5. The initial conditions for the new variables u, u_1, and u_2 can be found from the given initial conditions by means of system (13). As follows from Fig. 5, this method gives a setup requiring a large number of integrators in comparison with the method of direct integration. In addition, a comparatively large amount of time must be expended on the computation of the coefficients of the transformed equations.

3. Decomposition of the Initial Inhomogeneous nth-Order Equation into a System of n Inhomogeneous First-Order Equations

While this decomposition is not unique, the best results are given by the method described in [3], according to which the linear differential equation with constant coefficients

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_1\frac{dy}{dt} + a_0 y =$$

$$= b_0 x + b_1\frac{dx}{dt} + \ldots + b_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + b_n\frac{d^n x}{dt^n} \quad (14)$$

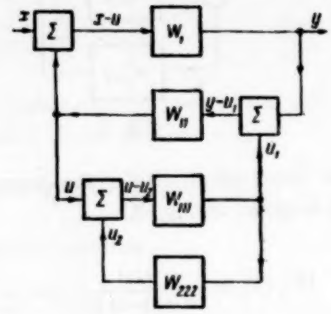

Fig. 4.

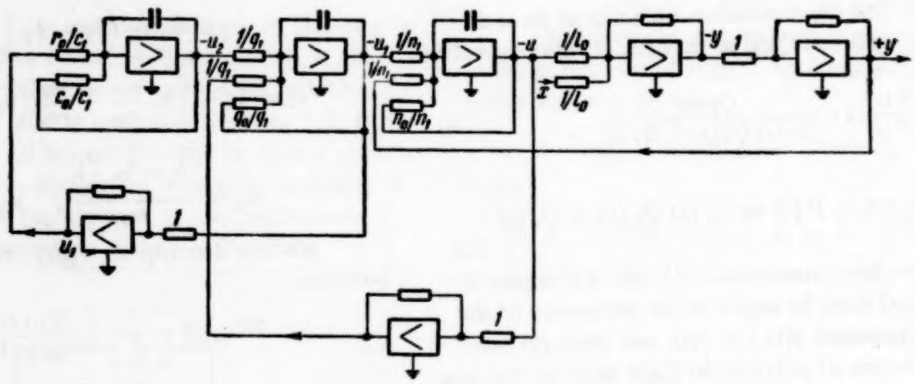

Fig. 5.

can be presented in the form of a system of linear first-degree differential equations:

$$\begin{aligned} y &= y_1 + \alpha_n x, \\ \frac{dy_1}{dt} &= y_2 + \alpha_{n-1}x, \\ &\ldots\ldots\ldots \\ \frac{dy_{n-1}}{dt} &= y_n + \alpha_1 x, \\ \frac{dy_n}{dt} &= -a_{n-1}y_n - a_{n-2}y_{n-1} - \\ &- \ldots - a_1 y_2 - a_0 y_1 + \alpha_0 x. \end{aligned} \quad (15)$$

Indeed, by eliminating $y_1, y_2, \ldots, y_n$ from the last equation of system (15), we arrive at an equation of the form

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_1\frac{dy}{dt} + a_0 y = \alpha_n\frac{d^n x}{dt^n} +$$

$$+ (\alpha_{n-1} + \alpha_n a_{n-1})\frac{d^{n-1}x}{dt^{n-1}} +$$

$$+ \ldots + (\alpha_1 + a_{n-1}\alpha_2 + \ldots + a_1\alpha_n)\frac{dx}{dt} +$$

$$+ (\alpha_0 + \alpha_1 a_{n-1} + \ldots + \alpha_{n-1}a_1 + \alpha_n a_0)\,x. \quad (16)$$

In order that Eqs. (15) and (16) be identical, it is necessary that the values of the new coefficients α_i satisfy the following equations:

$$\begin{aligned} b_0 &= \alpha_0 + \alpha_1 a_{n-1} + \alpha_2 a_{n-2} + \ldots + \alpha_{n-1}a_1 + \alpha_n a_0, \\ b_1 &= \qquad \alpha_1 + \alpha_2 a_{n-1} + \ldots + a_{n-1}a_2 + \alpha_n a_1, \\ b_2 &= \qquad\qquad \alpha_2 + \ldots + \alpha_{n-1}a_3 + \alpha_n a_2, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ b_{n-1} &= \qquad\qquad\qquad \alpha_{n-1} + \alpha_n a_{n-1}, \\ b_n &= \qquad\qquad\qquad \alpha_n. \end{aligned} \quad (16a)$$

The block schematic for the system of equations (15) is constructed in the form shown on Fig. 6a. The total number of blocks required is n + 3. The coefficients α_0, $\alpha_1, \ldots, \alpha_n$ are easily computed on the basis of (16a) by successively substituting the values of α_1, starting with $\alpha_n = b_n$. The initial conditions for the new variables y_1, $y_2, \ldots, y_n$ are determined by means of system of equations (15) from the given $y(0), y^{(1)}(0), \ldots y^{n-1}(0)$

and $\quad x(0),\ x^{(1)}(0),\ x_0^{(2)}(0), \ldots, x^{(n-1)}(0).$

As an example, we consider the reproduction of the transfer function

$$W(s) = \frac{s^2 - \frac{6}{\tau}s + \frac{12}{\tau^2}}{s^2 + \frac{6}{\tau}s + \frac{12}{\tau^2}},$$

approximating the transfer function of a lag link, e^{-sT}, by a Padé series for $\mu = \nu = 2$ [6]. Here, n = 2, $a_0 = 12/\tau^2$, $a_1 = 6/\tau$, $a_2 = 1$, $b_0 = 1/\tau^2$, $b_1 = -6/\tau$, and $b_2 = 1$.

On the basis of (15), the equivalent system of equations is written in the form

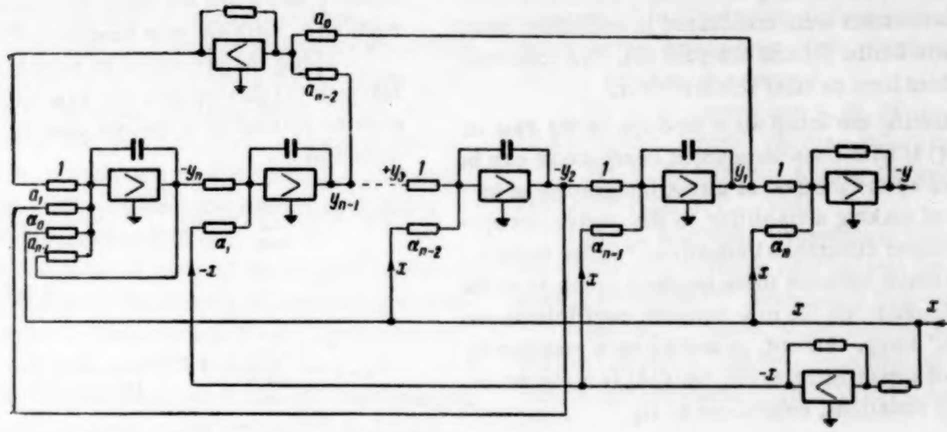

Fig. 6a.

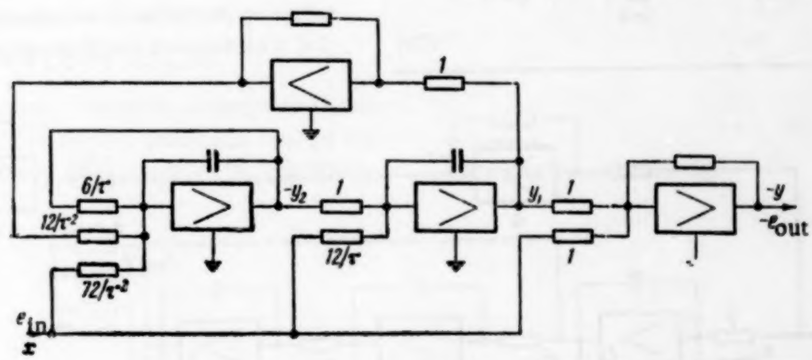

Fig. 6b.

$$\frac{dy_2}{dt} = -a_1 y_2 - a_0 y_1 + \alpha_0 x,$$

$$\frac{dy_1}{dt} = y_2 + \alpha_1 x, \quad y = y_1 + \alpha_2 x,$$

where $\underline{y}$ corresponds to the output signal e_{out} and $\underline{x}$ to the input signal e_{in}. Based on what has gone before, the coefficients α_0, α_1 and α_2 will be

$$\alpha_2 = b_2 = 1, \quad \alpha_1 = b_1 - a_1 = -\frac{12}{\tau}, \quad \alpha_0 = \frac{72}{\tau^2}.$$

The setup is shown on Fig. 6 b.

4. The Method of Combining Derivatives

We separate initial Eq. (14) into two by introducing the new variable

$$u = \frac{x}{p^n + a_{n-1}p^{n-1} + \ldots + a_1 p + a_0}. \tag{17}$$

As the result, we get

$$y = b_n \frac{d^n u}{dt^n} + b_{n-1}\frac{d^{n-1}u}{dt^{n-1}} + \ldots + b_1 \frac{du}{dt} + b_0 u. \tag{18}$$

Expression (17) can be rewritten in differential form:

$$\frac{d^n u}{dt^n} + a_{n-1}\frac{d^{n-1}u}{dt^{n-1}} + \ldots + a_1\frac{du}{dt} + a_0 u = x. \tag{19}$$

To establish the setup on the computer, it is initially necessary to "set up" Eq. (19) by the method of reducing the order of the derivatives and thereafter forming the desired variable $\underline{y}$ in the form of a sum of derivatives with respect to $\underline{u}$ with the proper coefficients. The values of the derivatives $d^i u / dt^i$ are obtained directly from the corresponding outputs of the integrators for the solution of Eq. (19). Some simplification of the setup is achieved if $d^n u / dt^n$ is eliminated from Eq. (18) by substituting its value from Eq. (19). The result is the set of equations

$$\frac{d^n u}{dt^n} + a_{n-1}\frac{d^{n-1}u}{dt^{n-1}} + \ldots + a_1\frac{du}{dt} + a_0 u = x,$$

$$y = (-b_n a_{n-1} + b_{n-1})\frac{d^{n-1}u}{dt^{n-1}} + \ldots + (-b_n a_1 + b_1)\frac{du}{dt} + (-b_n a_0 + b_0)u + b_n x.$$

For m = n = 3, the block schematic of the setup for these equations is shown in Fig. 7. In the general case, n + 3 operational blocks are needed in the setup. It is not necessary to carry out laborious computations to determine the coefficients for the setup.

5. Methods of Simulating Differential Equations with Variable Coefficients

Methods of simulating differential equations with variable coefficients were considered in sufficient detail by Laning and Battin [3] and Matyash [5]. We therefore limit ourselves here to brief remarks only.

Establishing the setup for a problem in the case of solving Eq. (14) with time-dependent coefficients can be implemented by the method of direct integration or by the method of making a transition to the equivalent system of first-order differential equations.* The fundamental difference between these methods amounts to the method of determining the new variable coefficients for the equations' setup. Indeed, in setting up a problem by the method of direct integration, Eq. (14) is replaced by the following equations, equivalent to it,

$$\sum_{j=0}^{n}(-1)^{j}(\alpha_j\, y)^{(j)} = -\sum_{j=0}^{m}(-1)^{j}(\beta_i x)^{(j)}, \tag{20}$$

where y and x are the same variables as in (14), but $\alpha_j(t)$ and $\beta_j(t)$ are new functions of time.

Using the properties of adjoint linear operators, Matyash [5] showed that the new variable coefficients must be related with the old ones by the following relationships:

$$\beta_{m-k} = \sum_{i=0}^{k}(-1)^{(m-i)}\frac{(m-i)!}{(m-k)!\,(k-i)!}\,b_{m-i}^{(k-i)}$$

$$(k = 0, 1, 2, \ldots, m),$$

$$\alpha_{n-k} = \sum_{i=0}^{k}(-1)^{n-i}\frac{(n-i)!}{(n-k)!\,(k-i)!}\,a_{n-i}^{(k-i)}$$

$$(k = 0, 1, 2, \ldots, n).$$

The block schematic for the setup of Eq. (20) is obtained by the methods used above for differential equation (14) with constant coefficients, with the sole difference that

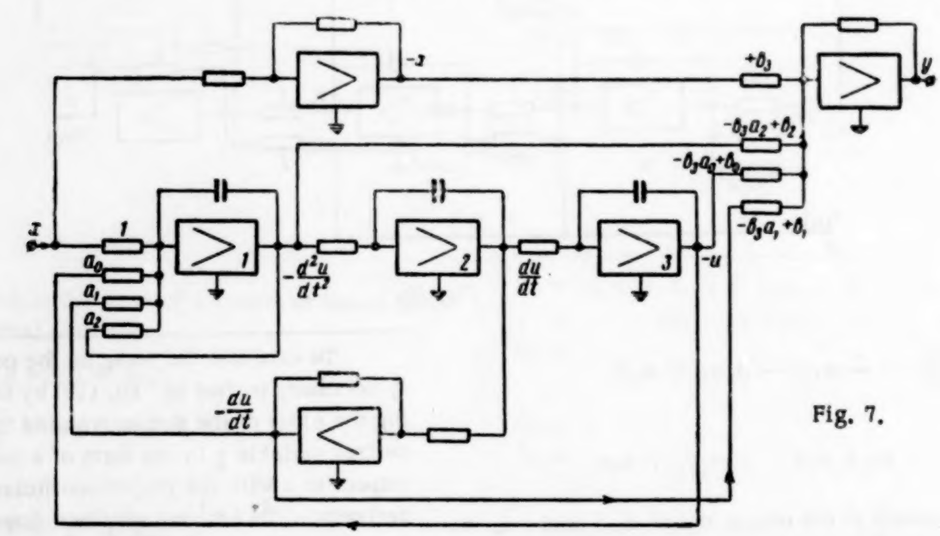

Fig. 7.

at the proper places, dividers of the variable coefficients are connected for the establishment of $\beta_{m-k}(t)$ and $\alpha_{n-k}(t)$.

The transition to an equivalent system of first-order differential equations is made exactly the same as in the case of constant coefficients, but here the new coefficients, $\alpha_n(t), \alpha_{n-1}(t), \ldots, \alpha_0(t)$, must be considered as some functions of time. The determination of these functions of time in terms of the original variable coefficients can be performed on the basis [3] of the recursion formulas

$$\alpha_0(t) = b_0(t),$$

$$\alpha_i(t) = b_i(t) - \sum_{k=0}^{i-1}\sum_{r=0}^{i-k} c_{n+r-i}^{n-i}\, a_{i-k+r}(t)\frac{d^r\alpha_k(t)}{dt^r}.$$

It follows from a comparision of the two methods considered that the transition to an equivalent system of first-order differential equations requires less computational work, since the original variable coefficients $a_j(t)$ enter into the setup in untransformed form. In solving a problem by the method of direct integration, one has to recompute all the variable coefficients entering into the equation actually set up.

SUMMARY

The comparison of the methods considered for simulating rational-fraction transfer functions allows the following conclusions to be drawn.

1. Reproduction of rational-fraction transfer functions and the differential equations which give rise to them by means of electronic analog computers without differentiating elements can be implemented by several methods. For nonzero initial conditions, one must also

* The methods of decomposing transfer functions into simpler ones and of combining derivatives are invalid here, since they lead to an interchange of differential operators which, with variable coefficients, is inadmissible.

know the values of the independent variable and its first (n-1) derivatives at the initial moment of time.

2. The minimum number of operational amplifiers in the block schematic of the setup is n + 3, where n is the order of the differential equation being set up, and does not depend on the setup method. An exception is the method of decomposing a transfer function into simpler ones, which leads to block schematics with larger numbers of operational amplifiers.

3. From the point of view of the amount of preliminary work required, the method of combining derivatives is the simplest. This method is only applicable to problems with constant coefficients.

4. For solving problems with variable coefficients, preference should be given to the method of making a transition to an equivalent system of first-order differential equations as requiring the least amount of additional computational work.

LITERATURE CITED

[1] A. K. Ganulich, "Methods of correcting distortions in the simulation of certain automatic control systems," [in Russian], Trudy Zhukovskii VVIA, 553 (1955).

[2] D. Mitchell, "The technology of using analog computes," in: Proceedings of the Session of Analog Computers in the Proceedings of the Conference at Cranfield in 1951 [Russian translation].

[3] J. H. Laning and R. H. Battin, Random Processes in Automatic Control [Russian translation] (IL, 1958).

[4] C. L. Johnson, Analog Computer Techniques (McGraw-Hill, N.Y., 1956).

[5] I. Matyash, "Programing linear differential equations with variable coefficients for their solution by analog computers," Avtomat. i Telemekh. 20, 7 (1959).†

[6] V. V. Gurov, V.M. Evseev, B. Ya. Kogan, A.A. Maslov, and F. E. Tranin, Type ÉMU-8 Electronic Analog Computer [in Russian] (Fil. VINITI, Moscow, 1957).

† See English translation.

ON CONSTRUCTING BRIDGE CIRCUITS BY THE SHORT-CIRCUIT METHOD

A. Ya. Tolchan

(Moscow)

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Net structure is proposed as the general representation of bridge circuits. On the basis of an analysis of complete nets, sufficient grounds for the possibility of constructing bridge circuits by the short-circuit method are deduced.

The problem of constructing, from an existent circuit's structural formula F(a,b,. . . , w), the bridge circuit equivalent to it is a very urgent one, since a significant gain in number of circuit elements might thereby be obtained in many cases. One of the methods of carrying out such constructions is well known under the name of the short-circuit method [1].

Practical application of this method is limited by the circumstance that, in the majority of cases, the parts of the circuit necessary for the construction are missing from the original structural formula F(a, b, . . . , w) [1].

In connection with this, the necessity arises of investigating the conditions under which use of this method is possible, as well as the methods of transforming the original structural formula to a form more convenient to use.

To investigate bridge circuits in the general case, it is convenient to present them in the form of a "complete net" as shown on Fig. 1. A complete net is a rectangular net with one element connected in each side of each mesh (elemental grid rectangle), where there are equal numbers of meshes in each file (row) in any given direction. All the nodes of its two opposite sides are connected to the input and output nodes.

The sides of the meshes are conventionally called longitudinal and lateral, depending on whether they are parallel or perpendicular to the input and output terminal buses. This nomenclature is also extended to the individual elements, depending on the side of the mesh they are connected in.

Longitudinal series of elements are called rows while lateral ones are called columns. Initial elements are denoted by I and final ones by F. The denotation of the remaining elements is in accordance with Fig. 1, and the numeration of rows and columns is from left to right and from top to bottom (not counting the rows of the initial and final elements).

The nodes of the net are denoted by $\alpha_{i,n}$, where $\underline{i}$ is the number of the longitudal elements' row and $\underline{n}$ the number of the lateral elements' column at the intersection of which the node is to be found.

Each planar bridge circuit can be converted (transformed) to a complete net. For this, all elements connected between two series-parallel nodes are replaced by one element; full rows and columns of longitudinal and lateral elements are then made up by adding elements equal to zero in places where gaps initially appeared, and elements equal to unity at places where conducting lines initially appeared. This transformation is not single-valued but among the variants of the resulting complete nets there is always a variant such that there are no zeros in the first or last rows of longitudinal elements. To obtain this variant, it suffices to begin the transformation by filling in these rows with elements occurring in the original planar scheme. Several examples of such a transformation are given in Fig. 2.

Conversely, from a net with a sufficient number of columns and rows of longitudinal and lateral elements, one may obtain any planar scheme, if part of the elements reduce to zero or to unity, and certain elements, where necessary, are replaced by series-parallel connections of a number of elements.

For nonplanar circuits, analogously, the general form of a bridge structure is a complete space lattice. Each of its facets (or of its cross sections parallel to any facet) is a complete planar net such as those considered above. Each cell of the lattice is a cube in each edge of which one element is connected. All the nodes of two opposite facets of the lattice are joined to the input and output nodes. Its conversion to a nonplanar circuit and vice versa are carried out by methods analogous to those described for plane nets.

All further exposition will deal exclusively with planar circuits and plane nets. The question of extending the conclusions derived from a consideration of plane nets to nonplanar circuits and to space lattices requires special consideration.

Relationships which exist in a complete net are valid in all cases when the individual elements equal zero or unity, i.e., are valid for any bridge circuit. Indeed, zero and unity are particular values which a circuit element may assume.

For the analysis of a complete net, we use a special "table of correspondences." Let $\underline{t}$ be a row of longitudinal elements and $\underline{s}$ be a column of lateral elements in a net.

Each column in a table is headed by one of the initial elements I_n, and each row by a final element F_p. In the cell of the table at the intersection of column I_n and row F_p, we write the full structural formula of the circuit joining nodes α_{1n} and α_{tp}, denoted by us as Π_{np}. As a preliminary step, this circuit must be presented (for ex-

ample, by a successive expansion by initial and final elements) in the form of a sum of circuits, each consisting of series connections of elements only. In the sequel, we shall call each such component circuit entering into Π_{np} an elementary circuit, and denote it by π_{np}.

Any bridge circuit presents itself, in the most general form, as some middle portion $f(a_{ij}, b_{mn})$ connected to input and output nodes via initial and final elements (Fig. 3 a). It is essential here that there be no isolated parallel circuits in the middle portion which are connected to the

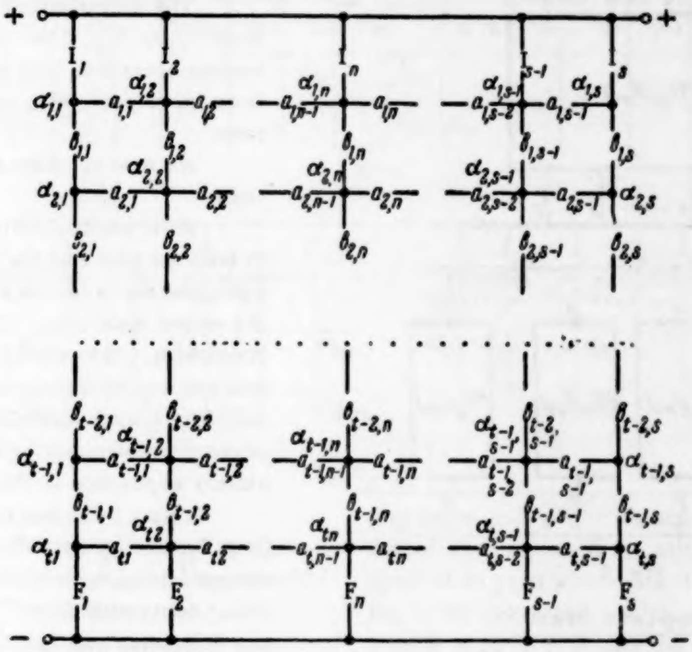


Fig. 1.

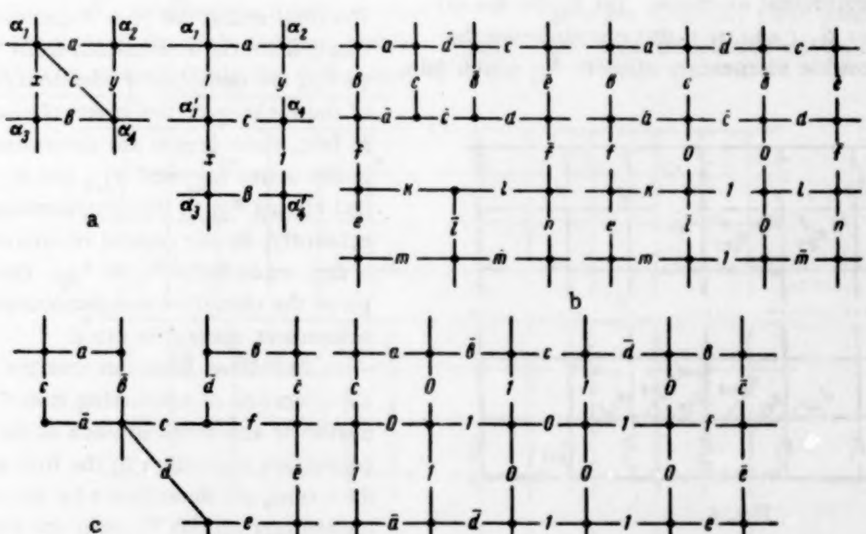


Fig. 2.

input and output nodes. This circumstance is used in the short-circuit method, where one initially seeks a middle portion of the circuit which is connected to the output node via all the final elements and is connected to the input node via one of the initial elements. After this, one seeks the points for connecting the remaining initial elements to the middle portion [1].

In the most general case, the decomposition of a bridge circuit by initial and final elements must give a set of parallel circuits, each of which is a middle portion, $f(a_{ij}, b_{mn})$ joined to the input node via one initial element and to the output node via one final element (Fig. 3 b).

This suggests that, among the elementary circuits into which any two of the full parallel circuits mentioned

above may be decomposed, there must be elements similar in content, and differing only in the placement of the points at which the initial and final elements are connected to the middle portions and the elements themselves.

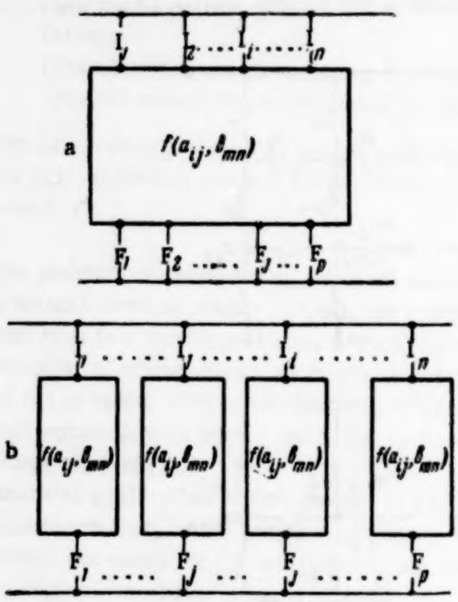


Fig. 3.

Analysis of a Complete Net

Let there be given some complete net, namely, a two-terminal network with s columns of lateral elements and t rows of longitudinal elements. Let Π_{ij} be the circuit joining nodes $\alpha_{1,i}$ and $\alpha_{t,j}$, this circuit being the sum of all the possible elementary circuits π_{ij} which join these nodes.

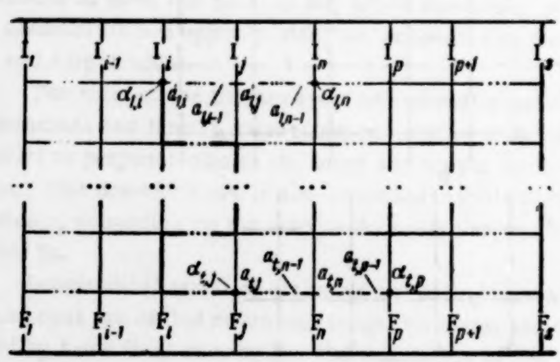


Fig. 4.

Theorem. If, in a complete net (Fig. 4), one choose any pair of nodes, $\alpha_{1,i}$ and $\alpha_{t,j}$, some other pair of nodes, $\alpha_{1,n}$ and $\alpha_{t,p}$, and then, in some elementary circuit π_{ij} of Π_{ij}, one replaces the elements in it which come from the set of longitudinal elements connected in the first row between nodes $\alpha_{1,i}$ and $\alpha_{1,n}$ and in the tth row between nodes $\alpha_{t,j}$ and $\alpha_{t,p}$ by all the elements of this set lacking in it, then the newly obtained elementary circuit will be some circuit π_{np} of Π_{np}.

On Fig. 4, only those nodes and elements mentioned in the theorem are noted. One of the possible circuits π_{ij} is shown in heavy lines.

In the sequel, we shall call such an operation the "operation of converting from π_{ij} to π_{np}" for two pairs of nodes from the first and tth rows. If this operation is carried out for all circuits π_{ij} of Π_{ij}, this is then the "operation of converting from Π_{ij} to Π_{np}".

The operations of converting from π_{ij} to π_{nj} and from π_{ij} to π_{ip}, these being particular cases of the conversion operation, may be considered as conversion operations in, respectively, only the first and only in the tth rows.

We now consider a number of preliminary propositions.

First, the conversion operation occurs analogously in both the first and the tth rows, since the structure of a complete net is identical on both the input node side and the output node side. This means that all conditions, relationships, and results of the conversion operation in the first row are identical to those in the tth row. We shall therefore always formulate propositions henceforth in terms of the first row, bearing in mind that they are also completely applicable to the tth row.

It may be shown that the operation of converting from π_{ij} to π_{np} actually breaks down into two independent operations, which can be carried out in either order: either conversion from π_{ij} to π_{nj}, i.e., in the first row, and thereafter from π_{nj} to π_{np}, i.e., in the tth row or, conversely, initially from π_{ij} to π_{ip}, i.e., in the tth row and then from π_{ip} to π_{np}, i.e., in the first row. Indeed, the final results for both sequences of operations are identical, since the conversions from π_{ij} to π_{nj} and from π_{ip} to π_{np} are completely identical, from the point of view of the makeup of the group of elements to be converted. In fact, these groups are determined only by the position of the nodes $\alpha_{1,i}$ and $\alpha_{1,n}$ and by the behavior of the initial circuit π_{ij} in the neighborhood of the first row which, evidently, do not depend on whether or not the conversion is first made from π_{ij} to π_{ip}. One can clearly use this to prove the complete independence of the order of the two component operations cited.

It follows from this that the character and result of the operation of converting from π_{ij} to π_{np} depend on the character and result of each of the individual component conversion operations in the first and tth rows which, in their turn, are determined by the behavior of the original elementary circuit π_{ij} near the first and the tth rows separately. This means that it is meaningful to differentiate the forms of the elementary circuits in the net by the character of their behavior close to the first (tth) row.

From this point of view, two cases are theoretically possible, while we shall formulate with respect to the first row only.

1. The original elementary circuit π_{ij} is so situated in the net that, once having started out from one node of the first row of longitudinal elements, it never again passes through any node of this row which is comprised in the set of nodes situated between $\alpha_{1,i}$ and $\alpha_{1,n}$, including

$\alpha_{1,n}$ (Fig. 5a,b,c). We remark that the elementary circuit can pass arbitrarily often through the remaining nodes of the first row.

2. The original elementary circuit π_{1j} is so situated in the net that, once having started out from one node of the first row of longitudinal elements, it again passes, at

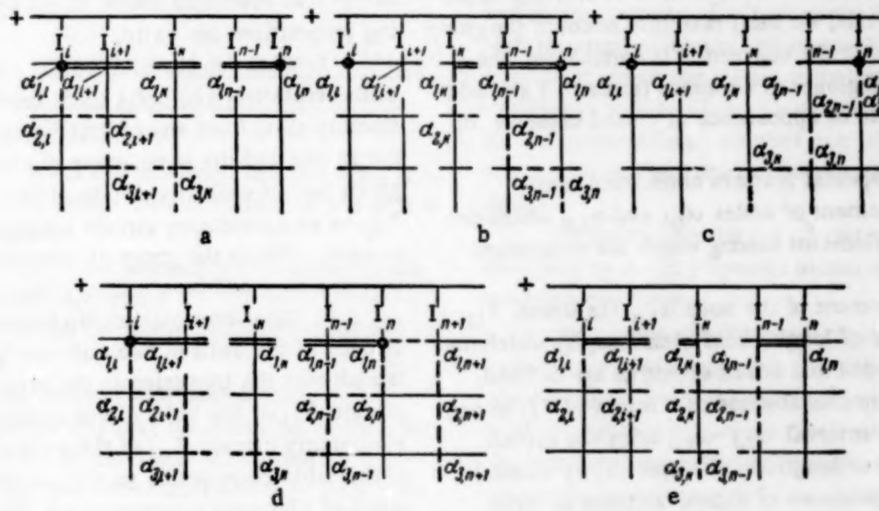


Fig. 5.

least once, through nodes of this row which are comprised in the set of nodes situated between $\alpha_{1,i}$ and $\alpha_{1,n}$, including $\alpha_{1,n}$ (Fig. 5 d,e).

We now show that, in these two cases, the conversion operation in the first row from π_{1j} to π_{nj} does in fact lead from π_{1j} to π_{nj}. The graphical proof of this is shown on Fig. 5 where, to avoid encumbering the figure, no elements are shown in the sides of the net's meshes. These latter are understood to be present. Only the nodes occurring in the path of the elementary circuit are notated. The dashed lines represent the initial elementary circuit, while those of its parts which pass through elements participating in the conversion operation are shown by solid lines. Double solid lines denote the circuit paths through the remaining elements in the group subject to conversion. Thus, the results of this operation are clearly visible.

There are obviously three possible diverse circuits for the first case, mentioned above, of the behavior of the

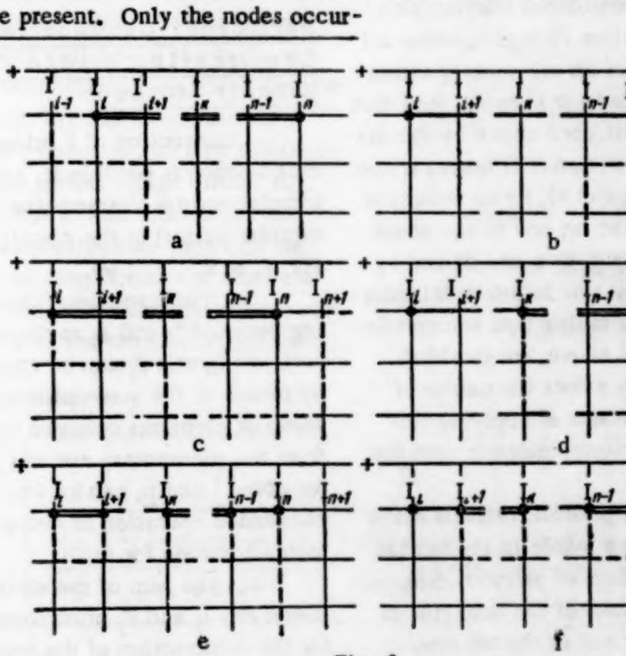


Fig. 6.

original elementary circuit in the neighborhood of the first row. The conversion operations for them are treated in Fig. 5 a,b, and c, from which it is clear that these operations indeed lead to π_{nj}.

We must consider two diverse circuits (Fig. 5d and e) for the second case cited above, finding that in both of them the conversion operation leads to π_{nj}, although with the addition of a closed contour.

In order to extend this proof to the general case of the theorem, we first consider which special features of the behavior of the original circuit in the net affect the course and the result of the operation of converting from π_{ij} to π_{nj}. With this, we shall take into account the group of elements in which the conversion is carried out, the make-up of the subgroups of elements (discarded and added) and the absence or appearance of closed contours in series with π_{nj}.

Among the special features considered are:

1) the placement of nodes $\alpha_{1,i}$ and $\alpha_{1,n}$ which define the group of elements among which the conversion is carried out;

2) the placement of the node $\alpha_{1,k}$ via which π_{ij} leaves the first row of longitudinal elements, by which the subgroups of discarded and added elements are defined;

3) the presence or absence of a return of π_{ij} to one of the nodes in the interval $\alpha_{1,i}-\alpha_{1,n}$ after the circuit leaves the first row of longitudinal elements, by which is determined the appearance of closed contours in series with π_{nj}.

It is also essential that the behavior of π_{ij} in the middle portion of the net in no way affects the course of the operation of conversion from π_{ij} to π_{nj} and, as was stated earlier, that this operation occurs independently in the first and the _t_th rows.

By taking these data into account, we conclude that all the possible variants of the behavior of an elementary circuit in the neighborhood of the first row, from the point of view of the character of the conversion operation and its results, lead to the two cases considered above. This becomes obvious from a consideration of Fig. 6, where all possible variants of the behavior of an elementary circuit close to the first row are represented. It is easily seen that they differ from the two cases considered above by the direction of the circuit's path in the net after it leaves a node in the interval $\alpha_{1,i}-\alpha_{1,n}$ (Fig. 6 a and b), by its returns to the first row of longitudinal elements at one of the nodes outside of the interval $\alpha_{1,i}-\alpha_{1,n}$ (Fig. 6 c and d) and by the number of its returns to the first row in the nodal interval $\alpha_{1,i}-\alpha_{1,n}$ (Fig. 6 e and f). By taking into account the essential special features presented above, we establish that all these differences in no way affect the course of the conversion operation, and the result of applying this operation to them will also be in correspondence with the theorem given above.

Finally, it is obvious that all possible variants of the course of the elementary circuit as a whole in the net can be given as the different combinations of pairwise different variants of these two general cases of the behavior of a circuit in the vicinity of the first and of the _t_th row, in conjunction with an arbitrary behavior in the remaining portion of the net. But, since the theorem is valid for all variants of these two cases, it is then valid for all forms of elementary circuits which are possible in a net.

Corollary. In presenting the corollary, we shall deal only with full nets, bearing in mind that it can be extended to any bridge circuit as a particular case of a net.

For some net, let a table of correspondences analogous to that described above be set up. Then, the following propositions are valid.

1. Between the elementary circuits of any two cells of the table I_iF_j and I_nF_p, there exists such an interrelationship that, from any circuit π_{ij} by a conversion carried out in one and the same group of elements common to the given pair of cells, we obtain either one of the circuits π_{np} or an elementary circuit longer than π_{np}, i.e., equal to zero. This is the group of elements connected between $\alpha_{1,i}$ and $\alpha_{1,n}$ and between $\alpha_{t,j}$ and $\alpha_{t,p}$.

2. The interrelationship between the elementary circuits of the cells of two columns I_i and I_n of this table is such that the transition in the rows from the elementary circuits π_{ij} of the cells of one column of the table to the elementary circuits π_{nj} of the cells of the other column of the table corresponds to the conversion in the one group of elements common to all the rows. In this group there appear the elements which are connected between $\alpha_{1,i}$ and $\alpha_{1,n}$ in the first row of longitudinal elements.

3. The connection between the elementary circuits of two rows F_j and F_p of the table is such that transition in the column from the elementary circuits of the cells of one row to the elementary circuits of the cells of the other row corresponds to the conversion in the group of elements which are common to all columns, this group containing elements connected between $\alpha_{t,j}$ and $\alpha_{t,p}$ in the _t_th row.

Sufficient Grounds for the Possibility of Constructing Bridge Circuits by the Short-Circuit Method

Construction of a bridge circuit by the short-circuit method is possible if, and only if, the following interrelationships characterize the individual elementary circuits (terms) in the circuit's original structural formula $F(I, F, a, b, \ldots, w)$.

1. The transition from the elementary circuits passing between I_i and F_j to the elementary circuits passing between I_n and F_j can be implemented, for given _i_ and _n_, by means of the conversion operation in one and the same group of elements common for any _j_, and the transition from the elementary circuits passing between I_i and F_p, for given _j_ and _p_, can be implemented by means of the conversion operation in one and the same group of elements common for any _i_.

2. The sum of the elementary circuits passing between any I_i and F_j must contain all the circuits necessary for the construction of the bridge circuit which is equivalent to this sum.

It is necessary to convince ourselves that the holding of these two conditions is sufficient for the construction of the bridge circuit by the short-circuit method. It is obvious that the holding of the second condition allows one to construct the bridge circuit of the middle portion $f(a_{ij},$

b_{mn}) (cf. Fig. 3 a), and the holding of the first condition guarantees the possibility of seeking such nodes in this middle portion that, upon their being connected to all initial and final elements, they will form the bridge circuit sought.

Practical Investigation of the Possibility of Constructing a Bridge Circuit by the Short-Circuit Method

Let there be given the structural formula F(a,b,..., w) of the circuit, the bridge circuit variant of which it is necessary to construct.

In order to verify the possibility of employing the short-circuit method, one must:

a) determine the complete group of initial and final elements;

b) set up the table of correspondences;

c) convince oneself that the make-up of the elements in all the cells of the table is identical;

d) verify that the aforementioned sufficient conditions hold, wherein the second condition is verified only for an arbitrary one of the table's cells.

If these conditions do not hold immediately for the original structural formula, one may then introduce, in the table's cells, additional elementary circuits which are equal to zero (containing mutually inverse elements or containing completely one of the pre-existing elementary circuits in the particular cell) and certain elements of the individual elementary circuits can be repeated.

When the necessary conditions have finally been met, the table obtained may be considered as the result of expanding the bridge circuit sought by initial and final elements where, in any cell I_iF_j of the table, are to be found all the elementary circuits by which the corresponding complete circuit Π_{ij} can be expanded.

Starting from this, we construct, for any arbitrarily chosen cell of the table, its partial bridge circuit Π_{ij}, which we then join to the initial and final nodes via I_i and F_j. On the circuit thus obtained we then find the nodes which, when the remaining initial and final elements are connected to them, allow the elementary circuits of the remaining cells of the table to be realized.

TABLE 1

	a_I	b_I	c_I
l_F	$b\bar{g}+ceh+e+\bar{e}h+\bar{g}h$	$eh+\bar{e}+a\bar{g}$	$egh+\bar{e}g+\bar{g}h$
m_F	$b+cg+e\bar{g}+h$	$e\bar{g}h+gh+\bar{e}\bar{g}$	h

This initial structural formula contains 42 elements in grouped form. Other variants of the grouping do not provide any significant gain in the number of elements necessary. After multiplying out the parentheses, we obtain a structural formula in the form of a sum of 18 elementary circuits.

However, the use of this methodology leads simply and rapidly to the desired result only in those rare cases when the bridge circuit sought contains no repeated elements and no mutually inverse elements. In the overwhelming majority of actual cases, bridge circuits contain both repeated elements and mutually inverse elements, as a result of which the initial structural formula lacks a significant portion of the elementary circuits necessary for the construction. Another part of these circuits is present in altered form due to the repetition of identical elements.

Therefore, for the use of this methodology, it is necessary to employ special means and an additional construction condition.

This latter condition is suggested by the fact that, generally speaking, there is no unique bridge circuit equivalent to some series-parallel circuit. However, in analogy with the usual requirement that a circuit be realized with the minimum number of elements, one can propose a universal requirement on the construction of bridge circuits, to wit, that there be a minimum number of repeated identical elements.

It seems reasonable to consider the methodology treated above, together with the additional means of construction, in terms of a concrete example.

Let there be given some structural formula for a circuit whose bridge variant must be constructed:

$$\begin{aligned} F = {} & a\{b(\bar{g}l + m) + c(ehl + gm) + \\ & + l[e + h(\bar{e} + \bar{g})] + m(e\bar{g} + h)\} + \\ & + b\{l(eh + \bar{e}) + m[h(e\bar{g} + g) + \bar{e}\bar{g}]\} + \\ & + c\{l[g(eh + \bar{e}) + \bar{g}h] + hm\} = ab\bar{g}l + \\ & + abm + acehl + acgm + ael + a\bar{e}hl + \\ & + a\bar{g}hl + ae\bar{g}m + ahm + behl + b\bar{e}l + \\ & + be\bar{g}hm + bghm + b\bar{e}\bar{g}m + \\ & + cegl + c\bar{e}\bar{g}l + c\bar{g}hl + chm. \end{aligned}$$

Without stopping to seek the initial and final elements, methods of finding which are described in detail in the literature [1], we state that the initial elements are a, b and c and that the final elements are l and m. We denote them by a_I, b_I, c_I, l_F, and m_F.

We now set up the table of correspondences (Table 1), writing in it successively the elementary circuits from the structural formula (without the initial and final elements).

A scan of the terms in the individual cells of the table by make-up of elements and make-up of individual terms shows that the bridge circuit being sought must contain both mutually inverse elements and repetitions. This means, that the construction problem now entails dealing with the minimum number of repeated elements and mutually inverse elements which are present in the original structural formula.

TABLE 2

	a_I	h	b_I	g	c_I
l_F	$e+\bar{e}h+\bar{g}ha+g\bar{g}hh_1+$ $+acgh_1$		$eh+\bar{e}+a\bar{g}+g\bar{g}h_1+$ $+aeghh_1$		$egh+\bar{e}g+\bar{g}h_1+aehh_1+$ $+ag\bar{g}$
$\bar{g}$					
m_F	$e\bar{g}+ha+\bar{e}\bar{g}h+ghh_1+$ $+aeg\bar{g}h_1$		$e\bar{g}h+\bar{e}\bar{g}+gh_1+a+$ $+aeg\bar{g}hh_1$		$h_1+ag+eg\bar{g}h+\bar{e}g\bar{g}+$ $+ac\bar{g}hh_1$

From this point of view, it follows from a comparison of terms in the cells $a_I l_F$ and $a_I m_F$ with the remaining ones that the repeated elements $\underline{b}$ and $\underline{c}$ (b_I and b, c_I and c) must appear in the circuit being sought. It is easily remarked that the shift of the elementary circuits $ab\bar{g}l$, acehl, abm, and acgm to the cells, respectively, $b_I l_F$, $c_I l_F$, $b_I m_F$, and $c_I m_F$, i.e., taking as the initial elements in these circuits, not a_I, but b_I and c_I, allows one to deal with the repeated element a (a_I and a) only (Table 2). With this, the make-ups of the cell elements become identical (taking into account the possibility of repetition of the initial element of cells $a_I l_F$ and $a_I m_F$) except for $\underline{\bar{g}}$, which does not appear everywhere.

A comparision of cells $b_I l_F$ and $b_I m_F$ (Table 2), which contain the greatest numbers of terms, leads to the conclusion that the group of elements in which conversion is to be carried out for the cells of rows l_F and m_F consists of $\underline{\bar{g}}$.

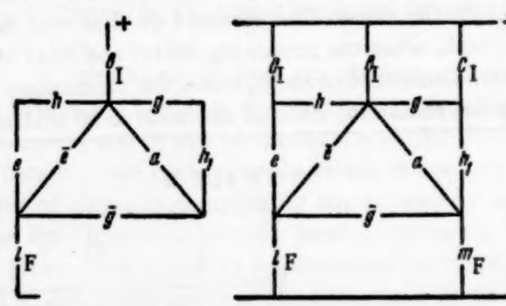
In the corresponding pairs of cells of rows l_F and m_F, we verify that this correspondence of terms holds, adding the deficiencies if they equal zero. A comparison of the cells of columns a_I and b_I leads to the acceptance of $\underline{h}$ as the sole element of the group in which conversion is carried out in columns a_I and b_I, with the condition that element $\underline{a}$ repeats, since otherwise a term equal to unity would appear in cell $b_I m_F$. Element $\underline{h}$ is also repeated (h and h_1) (Table 2).

In Table 2, all the added elementary circuits either contain mutually inverse elements or are longer ($a_I ghh_1 m_F$) than those already in the circuit ($a_I ahm_F$). In Table 2, all the added elements, circuits, and subscripts are given in boldface type.

Analogously, by considering the circuits of the table's cells $b_I l_F$ and $c_I l_F$, and $b_I m_F$ and $c_I m_F$, we find the most probable makeup of the group of elements to be converted, namely, g, for the cells of columns b_I and c_I.

We then fill all cells of the table until complete correspondence of all terms is attained. Now, the make-up of elements in the circuits of all the table's cells is the same for all, namely, a, e, $\bar{e}$, g, $\bar{g}$, h, and h_1. After this, we construct the equivalent bridge circuit for the elementary circuits of one of the table's cells, with the purpose of obtaining the middle portion of the bridge circuit being sought. In general, this construction could have been carried out earlier, without complete correspondence, i.e., when the first of the sufficient conditions for the possibility of constructing a bridge circuit was unfulfilled. This construction sometimes allows one to determine the make-up of the group of elements in which conversion is carried out. Thus, we consider the cell $b_I l_F$ of the table, since it contains the shortest elementary circuit. For this, we shall consider that elements $\underline{h}$ and $\underline{g}$ are connected in the first, and $\underline{\bar{g}}$ in the last, row of longitudinal elements, since they enter into the groups of elements in which conversion is carried out.

In the general case, this construction must be carried out by a method analogous to the one presented here.



a b

Fig. 7

However, in the majority of cases, the number of terms in the table's cells is small, and this construction can be carried out from the information obtained from all the cells' elementary circuits. Thus

$$f_{b_I l_F} = eh + \bar{e} + a\bar{g} + g\bar{g}h_1 + aeghh_1$$

and the equivalent bridge circuit is represented on Fig. 7 a. In it, one easily determines the nodes for connecting the

remaining initial and final elements, since one already knows the elements of the first and last rows of the circuit which are connected between these nodes. The final bridge circuit is shown on Fig. 7 b.

By considering the different elementary circuits of the scheme obtained, we convince ourselves that it is equivalent to the original structural formula F.

We note that there are, in the table, elementary circuits ($aegh_1$, $aeghh_1$, etc.) which consist of a somewhat shorter circuit from the same cell plus a closed contour (agh_1) abutting the first or last row of longitudinal elements. This is in complete agreement with the theorem proved earlier; such a circuit, naturally, does not violate equivalence. It should be added that, in seeking the group of elements in which conversion is to be carried out for the circuits of some pair of cells, it is ordinarily a good first move to consider a group which is contained completely in at least one term of each cell, and which is completely absent from a least one term.

To convince oneself of the truth of this regularity, it suffices to recall that among the elementary circuits of any cell there will always be a circuit of the form shown in Fig. 5 a, as well as a circuit of the form shown in Fig. 5 c.

When there are mutually inverse elements in this group, the rule given is false. The rule can also be violated in other cases of special locations of mutually inverse elements.

LITERATURE CITED

[1] M. A. Gavrilov, Switching Circuit Theory [in Russian] (Izdatelstvo AN SSSR, 1950).

NOISE STABILITY OF A FREQUENCY TELEMETERING SYSTEM FOR WEAK PULSE NOISE

Yu. I. Chugin

(Moscow)

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The noise stability of a telemetering frequency receiver is investigated for pulse noise of arbitrary duration. The threshold of noise stability and the optimal deviation frequency are determined.

The noise stability of the various forms of modulation have been most fully investigated (both theoretically and experimentally) for the case of fluctuation noise. Questions of noise stability for pulse noise, for frequency modulation in particular, have been treated with insufficient completeness in the literature.

In a number of the works devoted to this question, the treatment was mainly qualitative [1, 2]. There are works which contain incorrect initial assumptions [3] (as shown in [4]). There is definite interest inhering in the investigation of δ - function type noise in FM receivers which was given in [4, 5], but the action of such noise, so far from the actual noise encountered, very inadequately discloses the phenomena which occur in the receiver, and so the results thus obtained have a limited domain of applicability.

In the present paper we present the investigation of the effect of weak pulse noise of arbitrary duration on a frequency receiver. The results of the analysis are used for calculating the noise stability of a frequency telemetering system. The use of a narrow band output device allows one to use, as the criterion of noise stability of the telemetering system, the reduced mean-square error, defined as the ratio of the effective noise voltage V_n at the receiver's output to the maximum voltage of the output signal obtained by varying the signal frequency from $-\omega_d$ to $+\omega_d$, where ω_d is the maximum deviation frequency. The noise voltage V_n can be determined by the energy spectrum of the pulse noise at the discriminator's output. If a sequence of random noise pulses act at the receiver's input, then the phase φ_0 of the carrier frequency at the moment when a noise pulse acts can be considered a random variable, and the result of the noise's action at the discriminator's output can be considered a random pulsed process.

1. Reduced Mean-Square Error at the Receiver's Output

By definition, the magnitude of the reduced error at the output of the frequency receiver (Fig. 1) with pulse noise equals

$$\delta = \frac{V_n}{2U_{max}}, \tag{1}$$

where U_{max} is the maximum voltage of the receiver's output signal, equal to ω_d (for an ideal discriminator with characteristic slope of $s = 1$), V_n is the noise voltage at the receiver's output, defined as $V_n = \sqrt{P_n}$, where

$$P_n = \frac{1}{2\pi}\int_0^\infty W(\omega) K(\omega)\, d\omega \tag{2}$$

is the noise power at the receiver's output, $W(\omega)$ is the energy spectrum of the sequence of random noise pulses at the discriminator's output and $K(\omega)$ is the frequency characteristic of output filter F_2.

We now determine δ for an ideal and for a bell-shaped form of the output filter's frequency characteristic.

a. The Ideal Filter Form

By substituting in (2) the value of $W(\omega)$ obtained in Appendix II (formula II.16) and then integrating, we obtain the noise power at the output of a filter with pass band 0 to F_f and $K(\omega) = 1$:

$$P_n = 0.425\frac{m\omega_f^3}{\omega_0^2}\left(\frac{U_n}{U_s}\right)^2 \Phi_I. \tag{3}$$

Here,

$$\Phi_I = 1 - 3\left[\frac{2}{\alpha^2}\cos\alpha + \left(\frac{1}{\alpha} - \frac{2}{\alpha^3}\right)\sin\alpha\right] \times$$
$$\times \cos(\omega_0 + \lambda\omega_d)\tau -$$
$$-\frac{3\pi}{2}\left(\frac{\lambda\omega_d}{\Delta\omega}\right)\left[\left(-1 + \frac{6}{\alpha^2}\right)\times\right.$$
$$\left.\times\cos\alpha + 3\left(\frac{1}{\alpha} - \frac{2}{\alpha^3}\right)\sin\alpha\right]\frac{\sin(\omega_0 + \lambda\omega_d)\tau}{\Delta\omega\tau},$$

U_s is the amplitude of the sinusoidal signal at the receiver's input, U_n is the amplitude of the noise pulse at the receiver's input, τ is the duration of the noise pulse, m is the average number of pulses per second, λ is the transmitted parameter, varying from -1 to $+1$, ω_0 is the mean frequency of the input filter, $\alpha = \omega_f\tau$, $\omega_f = 2\pi F_f$, and $\Delta\omega = 2\pi\Delta f$.

By substituting the values of $V_n = \sqrt{P_n}$ and U_{max} in formula (1), we obtain the expression for the reduced

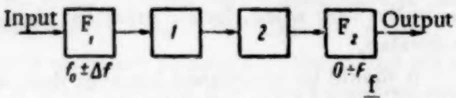


Fig. 1. Block schematic of the frequency receiver. F_1 is the input filter, 1 is a limiter, 2 is the discriminator, and F_2 is the output filter.

mean-square error at the output of a receiver with ideal output filter in the form

$$\delta_I = 0.33\,\frac{\sqrt{m}\,\omega_f^{3/2}}{\omega_d\omega_0}\,\frac{U_n}{U_s}\sqrt{\Phi_I}. \tag{4}$$

As follows from (II.17), the magnitude of the systematic error at the receiver's output equals zero.

On the basis of (I.12) and (I.13), formula (4) can be written in the form

$$\delta_I = 0.37\,\frac{\sqrt{m}\,\omega_f^{3/2}}{\omega_d\omega_0}\left(\frac{A_0}{U}\right)e^{-\frac{\pi}{8}\left(\frac{\lambda\omega_d}{\Delta\omega}\right)^2}\sqrt{\Phi_I}, \tag{5}$$

where A_0 is the maximum amplitude of the transient response at the output of the input filter to a voltage jump (or to a noise pulse of long duration, where the transient responses to the noise's leading and trailing edges do not overlap) and U is the signal amplitude at the filter's output.

The expression for Φ_1 is simplified if $\alpha \leq 2.5$. In this case,

$$\xi_1 = \frac{2}{\alpha^2}\cos\alpha + \left(\frac{1}{\alpha} - \frac{2}{\alpha^3}\right)\sin\alpha \approx \frac{1}{3} - \frac{1}{10}\alpha^2,$$

$$\xi_2 = \left(-1 + \frac{6}{\alpha^2}\right)\cos\alpha + 3\left(\frac{1}{\alpha} - \frac{2}{\alpha^3}\right)\sin\alpha \approx$$

$$\approx \frac{1}{5}\alpha^2 - \frac{1}{42}\alpha^4.$$

If $\alpha > 10$, then $\xi_1 \approx \sin\alpha/\alpha$ and $\xi_2 \approx -\cos\alpha + 3\sin\alpha/\alpha$. It is clear from formula (4) that the magnitude of the reduced error increases linearly with increasing amplitude of the noise pulses U_n and decreases with increasing mean signal frequency ω_0 and maximum deviation frequency ω_d. The error depends very little on the bandwidth of the input filter (the third term in the expression for Φ_I is small in comparison with unity). For $\lambda = 0$, $\delta \neq \varphi(\Delta f)$. The relationship of the error to the noise duration is expressed in a complicated manner. Formula (4), as follows from its derivation, is valid under the conditions when the maximum amplitude of the transient response to pulse noise at the input filter's output $B_{max}(t)$ does not exceed the amplitude of the signal $U(k_{max} = B_{max}(t)\ U \leq 1)$.

b. The Bell-Shape Filter Form

By carrying out the analogous computation for the bell-shaped form of the output filter's frequency characteristic

$$K(\omega) = e^{-\frac{\pi}{8}\left(\frac{\omega}{\omega_f}\right)^2}$$

we obtain the following formula:

$$P_n = \frac{2.3m\omega_f^3}{\omega_0^2}\left(\frac{U_n}{U_s}\right)^2\Phi_b, \tag{6}$$

where

$$\Phi_b = 1 - e^{-\frac{2}{\pi}\alpha^2}\left(1 - \frac{4}{\pi}\alpha^2\right)\cos(\omega_0 + \lambda\omega_d)\tau -$$

$$-\frac{\pi}{2}\left(\frac{\lambda\omega_d}{\Delta\omega}\right)e^{-\frac{2}{\pi}\alpha^2}\cdot\frac{4\alpha^2}{\pi}\left(3 - \frac{4\alpha^2}{\pi}\right)\frac{\sin(\omega_0 + \lambda\omega_d)\tau}{\Delta\omega\tau},$$

$$\delta_b = 0.76\,\frac{\sqrt{m}\,\omega_f^{3/2}}{\omega_d\omega_0}\left(\frac{U_n}{U_s}\right)\sqrt{\Phi_b}, \tag{7}$$

or

$$\delta_b = 0.85\,\frac{\sqrt{m}\,\omega_f^{3/2}}{\omega_d\Delta\omega}\left(\frac{A_0}{U}\right)e^{-\frac{\pi}{8}\left(\frac{\lambda\omega_d}{\Delta\omega}\right)^2}\sqrt{\Phi_b}. \tag{8}$$

2. Dependence of the Error on the Duration of the Noise Pulses

Figure 2 gives the square of the reduced mean-square error, computed from formulas (4) and (7) as a function of the noise duration for the ideal and bell-shaped output filter forms, respectively,

$$\phi_I = 0.33\,\frac{\sqrt{m}\,\omega_f^{3/2}}{\omega_d\omega_0}\left(\frac{U_n}{U_s}\right), \quad \phi_b = 2.3\phi_I, \quad \frac{\lambda\omega_d}{\Delta\omega} < 1.$$

It is clear from Fig. 2 that the curve for the relationship of the error and $\alpha = \omega_f\tau$ has the character of a modulated high-frequency oscillation. The high-frequency component depends on the relationship of the noise duration and the signal's frequency, i.e., on $\cos(\omega_0 + \lambda\omega_d)\tau$, and the envelope of the error variation curve depends on the relationship of the noise duration and the pass band of the output filter (on α).

As α increases, the error tends to a constant value, determined by the action on the receiver of the two pulse edges independently. The laws of variation of the error as a function of noise duration are similar for the ideal and the bell-shaped form of the output filter but, in the case of the bell-shaped form, the error's high-frequency component decays much more rapidly.

It follows from formulas (4) and (7) that, for a definite duration of the noise pulse [for $(\omega_0 + \lambda\omega_d)\tau = \pi$)], the magnitude of the error is a maximum, and equals

$$\delta_{I\,max} = 0.466\,\frac{\sqrt{m}\,\omega_f^{3/2}}{\omega_d\omega_0}\left(\frac{U_n}{U_s}\right), \tag{9a}$$

$$\delta_{B\,max} = 2.3\delta_{I\,max} \tag{9b}$$

For very long durations of the noise, when $\alpha \to \infty$ (in practice, even for $\alpha > 30$),

$$\delta_{I_{\alpha\to\infty}} = 0.33\,\frac{\sqrt{m}\,\omega_f^{3/2}}{\omega_d\omega_0}\left(\frac{U_n}{U_s}\right), \tag{10a}$$

$$\delta_{B_{\alpha\to\infty}} = 2.3\,\delta_{I_{\alpha\to\infty}}. \tag{10b}$$

It is clear from (9) and (10) that $\delta_{max} = \sqrt{2}\,\delta_{\alpha\to\infty}$. This relationship is self-evident since, by definition, $\delta \equiv \equiv \sqrt{P_n}$ and, as follows from physical considerations, $P_{n\,max} = 2P_{n\ \alpha\to\infty}$.

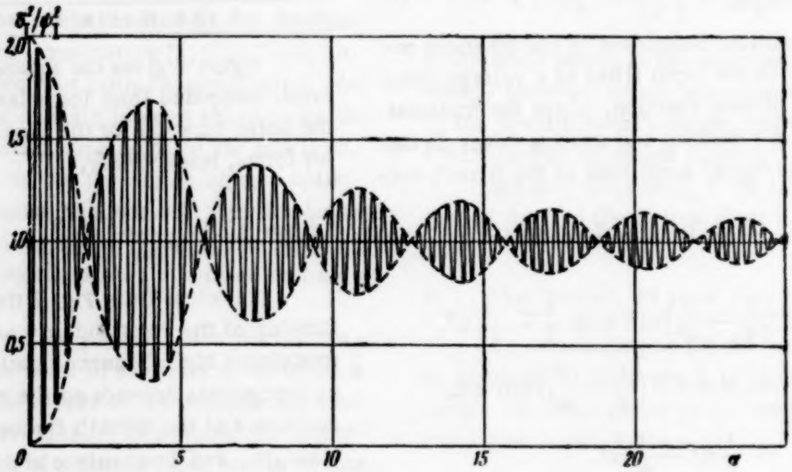
If the noise pulse has a duration much less than the period of the high-frequency carrier, i.e., if $\omega_0\tau \ll 1$, and the noise may be approximated by delta functions, then

$$\delta_{I_{\omega_0\tau<1}} = 0.233\,\frac{\sqrt{m}\,\omega_f^{3/2}}{\omega_d}\left(\frac{U_n\tau}{U_s}\right), \qquad (11a)$$

$$\delta_{b_{\omega_0\tau<1}} = 2.3\,\delta_{I_{\omega_0\tau<1}}. \qquad (11b)$$

As follows from Fig. 2 and from formulas (4) and (7), the influence of the phase relationships (of the high frequency) on the magnitude of the error exists, not only in the case when the transient responses from the leading and trailing edges of the noise pulse overlap [as follows from (1.13) for $\sqrt{\beta}\tau < 4$], but also in the case when these transient responses have a significant separation between them, i.e., in the case when, for all practical purposes, they do not overlap.

It should be mentioned however that, in actual conditions, due to the instability of the duration of the noise pulses (this instability being of significant magnitude when the durations are long), an averaging of the error within the limits of variation of the noise pulses $\Delta\tau$ can occur and for $\Delta\tau > (0.5/f_0)$, the averaged error will be close to $\delta_{\alpha\to\infty}$. In practice, one frequently has to do with noise pulses of unknown duration. In these cases, it is most reasonable to carry out the computations for the worst case (i.e., for that duration of the noise which maximizes the error). From formulas (9a) and (9b), one can determine the maximum error (for the critical τ) which is possible when the full bandwidth of the input filter is used, when $\omega_d = \Delta\omega - \omega_f = \text{const}$, with the conditions that the transient responses from the individual noise pulses at the output of filter F_1 do not overlap and that $m \to 2\Delta f$ and



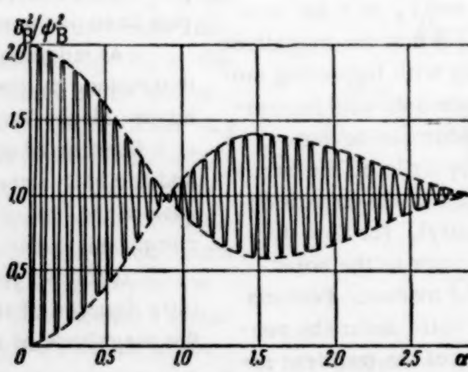


Fig. 2.

$k_{max} \to 1$. If these conditions hold (cf. II.15), formulas (9a) and (9b) will have the form:

$$\delta^{*}_{I\,max} \leqslant \frac{0.1}{(\gamma-1)\sqrt{\gamma}}, \qquad (12a)$$

$$\delta^{*}_{b\,max} \leqslant \frac{0.23}{(\gamma-1)\sqrt{\gamma}}, \qquad (12b)$$

where $\gamma = \Delta\omega/\omega_f = \Delta f/F_f$. For telemetry systems, one ordinarily has $\gamma \geq 5$ and, as follows from (12), $\delta^{*}_{I\,max} \leq 1.1\%$ and $\delta^{*}_{B\,max} \leq 2.5\%$. The maximum error obtainable in the very worst conditions, cited above, is thus relatively small.

It is clear from the formulas given above that the error quantities δ_{max}, $\delta_{\alpha\to\infty}$, $\delta_{\omega_0\tau\ll1}$ and δ^{*}_{max} are

2.3 times greater for the bell-shaped form of the output filter than for the case of the ideal filter form.

It is necessary to note that the taking off experimentally of the function $\delta = \varphi(\tau)$ gives rise to technical difficulties related to the necessity of a high stability of τ, particularly for large values of f_0 but, nonetheless, the character of the curves of Fig. 2 was substantiated experimentally.

3. Noise Stability Threshold for Noise Pulses

As was stated earlier, the formulas for the reduced mean-square error are valid for $k_{max} = B_{max}(t)/U \le 1$. Since the theoretical analysis of the case when $k_{max} > 1$ gives rise to mathematical difficulties, noise stability under these conditions was investigated experimentally. Figure 3 shows the function $\delta_{max} = \varphi(k_{max})$, taken off experimentally for the following values of the parameters: $\tau = 15.5$ msec, $m = 17$, $f_0 = 1$ kc, $2\Delta f = 150$ cps, $F_f = 2$ cps, $\gamma = 0$. The dashed line gives the same relationship computed from the theoretical formula (9b). It is clear from Fig. 3 that, for $k_{max} < 1$, the error is of small magnitude (less than 0.1%) and increases linearly with increasing k_{max}, a rapid growth in the error occurring for $k_{max} > 1$. The experimental results bear out the assertion of [1] that the condition $k_{max} = 1$ corresponds to the threshold of noise stability for frequency modulation for pulse noise. For the condition to hold that $k_{max} \le 1$ (the equal sign corresponds to the noise stability threshold), the following conditions must be met [cf. formulas (II.13)-(II.15)]:

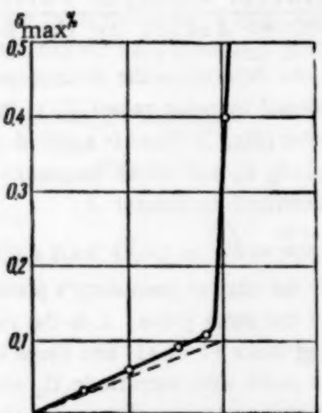


Fig. 3.

1) For the duration of the noise pulses much less than one period of the high frequency, i.e., for $\omega_0\tau \ll 1$,

$$\frac{U_n}{U_s}\Delta\omega\tau \leqslant 0.75;$$

2) for virtual nonoverlapping of the transient processes from the leading and trailing edges of the individual noise pulses, i.e., for $\Delta f\tau > 0.8$,

$$\frac{U_n}{U_s}\frac{\Delta\omega}{\omega_0} \leqslant 0.75;$$

3) for overlapping of the transient responses from the edges of the noise pulses and a doubled total reaction,

$$\frac{U_n}{U_s}\frac{\Delta\omega}{\omega_0} \leqslant 0.375.$$

4. Optimal Deviation Frequency for Pulse Noise

Figure 4 shows the curves for the relationship of the maximum error (for the critical duration of the noise pulses)

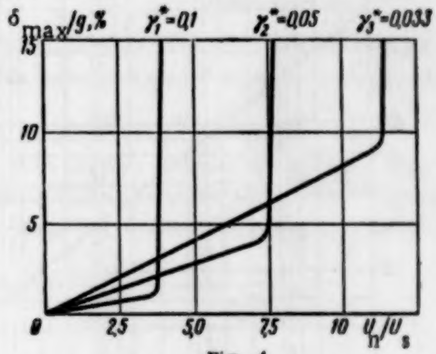


Fig. 4.

and the ratio U_n/U_s at the receiver input for various values of $\gamma = \Delta f/F_f$, constructed on the basis of theoretical (with $k_{max} < 1$) and experimental (for $k_{max} > 1$) data (the sharp bends in the curves correspond to the condition $k_{max} = 1$):

$$g = \frac{17.5\sqrt{m}\,\omega_f^{1/2}}{\omega_d\Delta\omega}, \quad \gamma^* = \gamma\frac{\omega_f}{\omega_0}.$$

It is clear from Fig. 4 that, if it be required to provide a small error for a large amount of noise (small δ_{max}/g), then it is advisable to work with large values of γ, since this error will be attained for the greatest amount of noise.

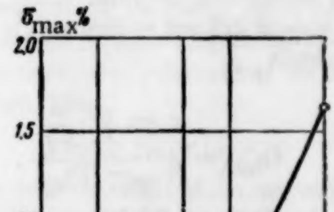


Fig. 5.

However, if δ_{max}/g is large, it is then more advantageous to reduce the input filter's bandwidth (reduce γ). Thus, for constant U_n/U_s, there exists an optimal pass band of the input filter and, consequently, an optimal deviation frequency for which the mean-square error at the receiver's output will be minimal. The determination of the conditions corresponding to the optimal deviation

frequency was carried out experimentally. Figure 5 shows the function $\delta_{max} = \varphi(2\Delta f)$, obtained experimentally for τ = 35.5 msec, m = 17, f_0 = 1 kc, F_f = 2 cps, and k_{max} = = 1 for $2\Delta f$ = 80 cps. The dashed line gives the theoretical function, constructed from formula (9b). It is obvious from Fig. 5 that the minimum error occurs for k_{max} = 1. The value of the optimum pass band can thus be defined from expressions (II.13)-(II.15).

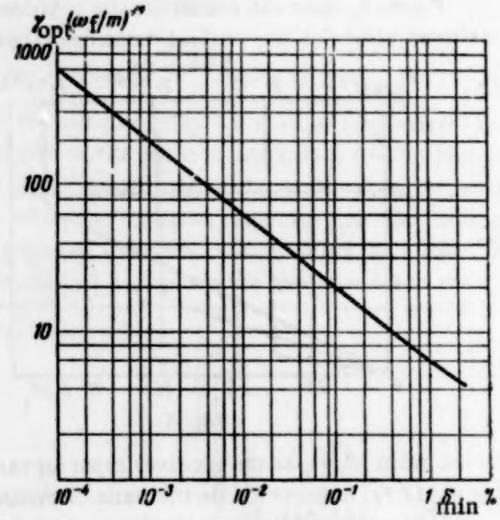


Fig. 6.

The optimal deviation frequency will equal $\omega_{d\,opt} = \Delta\omega_{opt} - \omega_f$ and, for small ω_f, $\omega_{d\,opt} \approx \Delta\,\omega$. The presence of the optimal deviation frequency for k_{max} = 1 also follows from the curves of Figs. 3 and 4, if one takes into account the sharply increasing error for $k_{max} > 1$.

Knowing $\omega_{d\,opt}$, one can determine the least value of error possible for a given ratio U_n / U_s at the receiver input. By substituting the value of $\omega_{d\,opt} = \Delta\,\omega_{opt} - \omega_f$, where $\Delta\,\omega_{opt}$ is defined by formula (II.15), in (9a) and (9b), we obtain

$$\delta_{I\,min} = \frac{0.175\sqrt{\frac{m}{\omega_f}}}{(\gamma_{opt}-1)\,\gamma_{opt}}, \tag{13a}$$

$$\delta_{b\,min} = 2.3\,\delta_{I\,min}. \tag{13b}$$

If the number of noise pulses per second $m \to 2\Delta f$, then formulas (13) coincide with formulas (12). Figure 6 shows γ_{opt} as a function of δ_{min}, obtained from formula (13b). It is clear that, for the choice of the optimal pass band for which the maximum range of the action of the telemetering system is provided, it suffices to know the system's error, its speed of response (the quantity ω_f), and the average number of noise pulses per unit time. After γ_{opt} has been chosen, it is necessary to verify the condition of nonoverlap of the transient responses from the individual noise pulses at the input filter's output, since the formulas obtained are rigorously valid only for this condition. The question of the receiver's noise stability when this condition of nonoverlap is violated requires additional investigation.

SUMMARY

1. With operation of a frequency receiver of a telemetering system under conditions of weak noise pulses of arbitrary duration (without special noise-protection measures having been taken), when the maximum amplitude of the transient response to the noise does not exceed the signal amplitude ($k_{max} < 1$), the reduced mean-square error at the receiver's output depends slightly on the input filter's bandwidth, decreases with increasing maximum deviation frequency, and depends on the duration of the noise pulses in a complicated way.

2. With frequency modulation and noise pulses, there exist a threshold of noise stability and an optimal value of deviation frequency, both occurring when k_{max} = 1. The conditions under which $k_{max} \leq 1$ are defined by formulas (II.13)-(II.15).

3. The maximum value of the error (for the critical duration of the noise pulses), obtained for complete employment of the input filter's bandwidth and with the condition of nonoverlap of the transient responses to individual noise pulses at the input filter's output and the conditions that $k_{max} \to 1$ and $\gamma \geq 5$, does not exceed 1.1% for an ideal characteristic of the output filter and 2.5% for a bell-shaped characteristic.

4. For $k_{max} > 1$, the error exceeds the admissible one for telemetry, and it is necessary to take special measures to limit the noise pulses.

APPENDIX I

Discriminator's Output Voltage with Noise Pulses Acting on the Receiver

We now determine the discriminator's output voltage engendered by noise pulses if, at the frequency receiver's input (Fig. 1) there is applied a sinusoidal signal with amplitude U_s and whose frequency depends linearly on the transmitted parameter λ:

$$u = U_s \cos\left[(\omega_0 + \lambda\omega_d)\,t + \varphi_0\right], \tag{I.1}$$

where φ_0 is the carrier frequency's phase at the moment of action of the noise pulse, λ is the transmitted parameter, varying from -1 to +1, and there simultaneously acts a noise pulse with amplitude U_n and duration τ. By placing the time origin at the center of the noise pulse, we can present it in the form of a difference of two dc voltage jumps, one of which acts at time $t = -\tau/2$, the other at time $t = \tau/2$. When one voltage jump acts on input filter F_1, the filter's output voltage can be given as

$$u_s = A(t)\cos\omega_0 t, \tag{I.2}$$

where A(t) is the voltage envelope of the transient response.

Using the principle of superposition for the action of two jumps, i.e., the noise pulse, we can write the filter's output voltage as

$$u_n = A\left(t+\frac{\tau}{2}\right)\cos\omega_0\left(t+\frac{\tau}{2}\right) - A\left(t-\frac{\tau}{2}\right)\cos\omega_0\left(t-\frac{\tau}{2}\right)$$

We give this expression in the form

$$u_n = B(t)\cos[\omega_0 t + \theta(t)]. \qquad (I.3)$$

where B(t) is the voltage envelop of the transient response engendered by the noise pulse of duration τ; $\theta(t)$ is the phase of the transient response voltage,

$$B(t) = \left[A^2\left(t+\frac{\tau}{2}\right) + A^2\left(t-\frac{\tau}{2}\right) - 2A\left(t+\frac{\tau}{2}\right)A\left(t-\frac{\tau}{2}\right)\cos\omega_0\tau\right]^{\frac{1}{2}}. \qquad (I.4)$$

$$\theta(t) = \operatorname{arctg}\frac{A\left(t+\frac{\tau}{2}\right)+A\left(t-\frac{\tau}{2}\right)}{A\left(t+\frac{\tau}{2}\right)-A\left(t-\frac{\tau}{2}\right)}\operatorname{tg}\frac{\omega_0\tau}{2}. \qquad (I.5)$$

The resulting voltage from signal and noise at the output of filter F_1 will equal

$$U_p = U\cos[(\omega_0+\lambda\omega_d)t+\varphi_0] + B(t)\cos[\omega_0 t+\theta(t)] = S\cos[\omega_0 t+\psi(t)], \qquad (I.6)$$

where U is the signal amplitude at the filter's output.

The phase of this oscillation is

$$\psi(t) = \operatorname{arc\,tg}\frac{U\sin(\lambda\omega_d t+\varphi_0)+B(t)\sin\theta(t)}{U\cos(\lambda\omega_d t+\varphi_0)+B(t)\cos\theta(t)}. \qquad (I.7)$$

The deviation of the frequency of the resulting oscillation from the mean frequency ω_0 is determined by the time derivative of the function $\psi(t)$, i.e., by the magnitude of $d\psi/dt$, and the deviation of the frequency of this oscillation from the signal's frequency (equal to $\omega_0 + \lambda\omega_d$) is defined by the expression $\Delta\omega_r(t) = d\psi/dt - \lambda\omega_d$.

By differentiating, we obtain

$$\Delta\omega_r(t) = \frac{-\frac{dk}{dt}\sin\gamma}{1+k^2+2k\cos\gamma} - \left(\lambda\omega_d - \frac{d\theta}{dt}\right)\frac{k^2+k\cos\gamma}{1+k^2+2k\cos\gamma}, \qquad (I.8)$$

where

$$k = k(t) = \frac{B(t)}{U}, \quad \gamma = \lambda\omega_d t + \varphi_0 - \theta(t).$$

For $k < 1$, the fractions in (I.8) can be expanded in simple series:

$$\frac{\sin\gamma}{1+k^2+2k\cos\gamma} = \sum_{n=1}^{\infty} k^{n-1}\sin n\gamma,$$

$$\frac{k^2+k\cos\gamma}{1+k^2+2k\cos\gamma} = \sum_{n=1}^{\infty} k^n\cos n\gamma.$$

With this, formula (I.8) is written in the form

$$\Delta\omega_r(t) = (-1)^n\left\{\frac{dk}{dt}\sum_{n=1}^{\infty}k^{n-1}\sin n\gamma - \left(\lambda\omega_d - \frac{d\theta}{dt}\right)\sum_{n=1}^{\infty}k^n\cos n\gamma\right\}. \qquad (I.9)$$

By assuming an ideal frequency discriminator, i.e., one whose output voltage is directly proportional to the deviation of the frequency from the mean value ω_0:

$$u_{out.d} = s(\omega-\omega_0), \qquad (I.10)$$

we get from (I.9) that the discriminator's output voltage, when noise pulses act on the receiver's input, will equal

$$u_{out}(t) = s\Delta\omega_r(t). \qquad (I.11)$$

In what follows, we shall assume the slope of the discriminator's characteristic to be $s = 1$. For the case of a bell-shaped frequency characteristic $C(\omega)$ of input filter F_1

$$C(\omega) = C_0 e^{-\frac{\pi}{2}\frac{(\omega-\omega_0)^2}{(2\Delta\omega)^2}}, \qquad (I.12)$$

where $\Delta\omega = 2\pi\Delta f$ and $2\Delta f$ is the bandwidth; the voltage envelope of the transient response at the filter's output when a dc voltage of magnitude U_n acts at the input is determined by the formula [6]

$$A(t) = A_0 e^{-\beta t^2} = \frac{\sqrt{8}}{\pi}U_n C_0\frac{\Delta\omega}{\omega_0}e^{-\frac{2}{\pi}\Delta\omega^2 t^2}. \qquad (I.13)$$

APPENDIX II

Energy Spectrum of a Sequence of Noise Pulses at the Discriminator Output

The expression for the energy spectrum of the sequence of noise pulses appearing at the discriminator's output when random noise pulses act at the receiver's input has the form [7]:

$$W(\omega) = 2m|F(\omega)|^2, \qquad (II.1)$$

where $|F(\omega)|$ is the modulus of the spectral density of an individual noise pulse and m is the average number of pulses per second.

The spectral density of an individual noise pulse

$$F(\omega) = \int_{-\infty}^{\infty} u_{out}(t)e^{-i\omega t}dt \qquad (II.2)$$

can be written in the form

$$F(\omega) = A(\omega) - iB(\omega) = |F(\omega)|e^{-i\eta(\omega)}, \qquad (II.3)$$

where

$$A(\omega) = \int_{-\infty}^{\infty} u_{out}(t)\cos\omega t\,dt, \quad B(t) = \int_{-\infty}^{\infty} u_{out}(t)\sin\omega t\,dt, \qquad (II.4)$$

and $u_{out}(t)$ is defined by formula (I.11).

By substituting the value of $u_{out}(t)$ in (II.4) and computing the integrals, we obtain the expressions for the real and imaginary parts of the spectral density in the form

$$A(\omega) = \sum_{n=1}^{\infty} A_n(\omega) = \sum_{n=1}^{\infty} \frac{(-1)^n \omega}{n} \int_{-\infty}^{\infty} \left[\frac{B(t)}{U}\right]^n \sin n\,[\lambda\omega_d t + \varphi_0 - \theta(t)] \sin \omega t\, dt, \tag{II.5}$$

$$B(\omega) = \sum_{n=1}^{\infty} B_n(\omega) = \sum_{n=1}^{\infty} -\frac{(-1)^n \omega}{n} \int_{-\infty}^{\infty} \left[\frac{B(t)}{U}\right]^n \sin n\,[\lambda\omega_d t + \varphi_0 - \theta(t)] \cos \omega t\, dt.$$

By carrying out a number of trigonometric transformations, and by taking into account, as follows from (I.3), that

$$B(t)\cos\theta(t) = \left[A\left(t+\frac{\tau}{2}\right) - A\left(t-\frac{\tau}{2}\right)\right]\cos\frac{\omega_0\tau}{2},$$

$$B(t)\sin\theta(t) = \left[A\left(t+\frac{\tau}{2}\right) + A\left(t-\frac{\tau}{2}\right)\right]\sin\frac{\omega_0\tau}{2}, \tag{II.6}$$

we get, for $A(\omega)$ and $B(\omega)$, that

$$A(\omega) = \sum_{n=1}^{\infty} A_n(\omega) = \sum_{n=1}^{\infty} [A_{2n-1}(\omega) + A_{2n}(\omega)], \tag{II.7}$$

where

$$A_{2n-1}(\omega) = \frac{(-1)^{2n-1}\omega \sin(2n-1)\varphi_0}{2n-1} \left[\int_{-\infty}^{\infty} \left[\frac{B(t)}{U}\right]^{2n-1} \cos(2n-1)\theta \cos(2n-1)\lambda\omega_d t \sin\omega t\, dt + \right.$$

$$\left. + \int_{-\infty}^{\infty} \left[\frac{B(t)}{U}\right]^{2n-1} \sin(2n-1)\theta \sin(2n-1)\lambda\omega_d t \sin\omega t\, dt\right],$$

$$A_{2n}(\omega) = \frac{(-1)^{2n}\omega \cos 2n\varphi_0}{2n}\left[\int_{-\infty}^{\infty} \left[\frac{B(t)}{U}\right]^{2n} \cos 2n\theta \sin n\lambda\omega_d t \sin\omega t\, dt - \int_{-\infty}^{\infty} \left[\frac{B(t)}{U}\right]^{2n} \sin 2n\theta \cos 2n\lambda\omega_d t \sin\omega t dt\right],$$

$$B(\omega) = \sum_{n=1}^{\infty} B_n(\omega) = \sum_{n=1}^{\infty} [B_{2n-1}(\omega) + B_{2n}(\omega)], \tag{II.8}$$

where

$$B_{2n-1}(\omega) = -\frac{(-1)^{2n-1}\omega \cos(2n-1)\varphi_0}{2n-1} \left[\int_{-\infty}^{\infty} \left[\frac{B(t)}{U}\right]^{2n-1} \cos(2n-1)\theta \sin(2n-1)\lambda\omega_d t \cos\omega t\, dt - \right.$$

$$\left. - \int_{-\infty}^{\infty} \left[\frac{B(t)}{U}\right]^{2n-1} \sin(2n-1)\theta \cos(2n-1)\lambda\omega_d t \cos\omega t\, dt\right],$$

$$B_{2n}(\omega) = -\frac{(-1)^{2n-1}\omega \sin 2n\varphi_0}{2n-1}\left[\int_{-\infty}^{\infty} \left[\frac{B(t)}{U}\right]^{2n} \cos 2n\theta \cos 2n\lambda\omega_d t \cos\omega t\, dt + \int_{-\infty}^{\infty} \left[\frac{B(t)}{U}\right]^{2n} \sin 2n\theta \sin 2n\lambda\omega_d t \cos\omega t\, dt\right].$$

It follows from (II.7) and (II.8) that, after averaging over the phase angle φ_0 for its equiprobable values, one obtains the following valid relationships

$$\overline{[A(\omega)]^2} = \sum_{n=1}^{\infty} \overline{[A_n(\omega)]^2}, \quad \overline{[B(\omega)]^2} = \sum_{n=1}^{\infty} \overline{[B_n(\omega)]^2},$$

since $\overline{\sin n\varphi_0 \cos l\varphi_0} = 0$, $\overline{\sin k\varphi_0 \sin l\varphi_0} = 0$ and $\overline{\cos k\varphi_0 \cos l\varphi_0} = 0$.

With this, the energy spectrum of (II.1) can be written in the form

$$W(\omega) = 2m \sum_{n=1}^{\infty} \overline{|F_n(\omega)|^2}, \tag{II.9}$$

where $\overline{|F_n(\omega)|^2} = \overline{[A_n(\omega)]^2} + \overline{[B_n(\omega)]^2}$.

We now express the values of $B(t)^n \cos n\theta(t)$ and $B(t)^n \sin n\theta(t)$ in terms of their values for n = 1 (II.6), substitute them in the expressions for $A_n(\omega)$ (II.7) and $B_n(\omega)$ (II.8) and then integrate. In finding $[A_n(\omega)]^2$ and $[B_n(\omega)]^2$ and substituting in these expressions the values of A_0, β, and C_0 / U, defined in accordance with (I.12) and (I.13) by the formulas

$$A_0 = \frac{\sqrt{8} U_n C_0 \Delta\omega}{\pi\omega_0}, \quad \beta = \frac{2\Delta\omega^2}{\pi}, \quad \frac{C_0}{U} = \frac{e^{\frac{\pi}{8}\left(\frac{\lambda\omega_d}{\Delta\omega}\right)^2}}{U_s},$$

we get that the following real relationships hold for telemetering systems in which $\omega_{max} / \Delta\omega = \omega_f / \Delta\omega \leq 0.2$ and $\omega_d \leq \Delta\omega - \omega_f$:

$$|\overline{F_1(\omega)}|^2 = 4\left(\frac{\omega}{\omega_0}\right)^2\left(\frac{U_n}{U_s}\right)^2\left\{1 - \cos(\omega_0 + \lambda\omega_d)\tau\cos\omega\tau - \frac{\pi}{2}\left(\frac{\lambda\omega_d}{\Delta\omega}\right)\left(\frac{\omega}{\Delta\omega}\right)\sin(\omega_0 + \lambda\omega_d)\tau\sin\omega\tau\right\} \qquad \text{(II.10)}$$

$$|\overline{F_2(\omega)}|^2 = 4\left(\frac{\omega}{\omega_0}\right)\left(\frac{U_n}{U_s}\right)^2\left[\frac{1}{\pi^2}\left(\frac{\Delta\omega}{\omega_0}\right)^2\left(\frac{U_n}{U_s}\right)^2\right]\left\{1 - \cos 2(\omega_0 + \lambda\omega_d)\tau\cos\omega\tau + \right.$$

$$+ \frac{\pi}{2}\left(\frac{\lambda\omega_d}{\Delta\omega}\right)\left(\frac{\omega}{\Delta\omega}\right)\sin 2(\omega_0 + \lambda\omega_d)\tau\sin\omega\tau - 4e^{-\frac{\beta\tau^2}{2}}\left[\cos(\omega_0 + \lambda\omega_d)\tau\cos\frac{\omega\tau}{2} + \right.$$

$$\left.\left. + \frac{\pi}{2}\left(\frac{\lambda\omega_d}{\Delta\omega}\right)\left(\frac{\omega}{\Delta\omega}\right)\sin(\omega_0 + \lambda\omega_d)\tau\sin\frac{\omega\tau}{2}\right] + 2e^{-\beta\tau^2}\right\}, \qquad \text{(II.11)}$$

$$|\overline{F_3(\omega)}|^2 = 4\left(\frac{\omega}{\omega_0}\right)\left(\frac{U_n}{U_s}\right)^2\left[\frac{2.38}{\pi^4}\left(\frac{\Delta\omega}{\omega_0}\right)^4\left(\frac{U_n}{U_s}\right)^4\right]\left\{1 - \cos 3(\omega_0 + \lambda\omega_d)\tau\cos\omega\tau - \right.$$

$$\left. - \frac{\pi}{2}\left(\frac{\lambda\omega_d}{\Delta\omega}\right)\left(\frac{\omega}{\Delta\omega}\right)\sin 3(\omega_0 + \lambda\omega_d)\tau\sin\omega\tau\right\}. \qquad \text{(II.12)}$$

Formulas (II.10)-(II.12) are valid for $k_{max} = B(t)_{max}/U < 1$. We now determine the maximum value of $U_n \Delta\omega / U_s \omega_0$ for which the condition $k_{max} < 1$ holds.

The signal amplitude at the output of filter F_1

$$U = U_s C(\omega) = U_s C_0 e^{-\frac{\pi}{2}\frac{(\omega-\omega_0)^2}{(2\Delta\omega)^2}}$$

has its maximum value for $\omega - \omega_0 = \lambda\omega_d = \omega_d$. Since $\omega_d = \Delta\omega - \omega_f \approx \Delta\omega$ (for small ω_f), then $U_{min} = 0.677\, U_s C_0$.

If $\omega_0 \tau \ll 1$ then, as follows from (II.4), $B(t)_{max} = A_0 \omega_0 \tau$ and, for the condition $k_{max} < 1$ to hold, it is necessary that the following inequality hold

$$\frac{U_n}{U_s}\Delta\omega\tau < 0.75. \qquad \text{(II.13)}$$

If the transient responses from the leading and trailing edges of the individual noise pulses virtually do not overlap, which occurs, as follows from (I.13), for $\sqrt{\beta}\tau > 4$, then $B(t)_{max} = A_0$ and, in order for the inequality $k_{max} < 1$ to hold, it is necessary that the following condition be met

$$\frac{U_n}{U_s}\frac{\Delta\omega}{\omega_0} < 0.75. \qquad \text{(II.14)}$$

When the transient responses from the noise pulse edges do overlap, i.e., when $\sqrt{\beta}\tau < 4$ and when, for definite values of noise pulse duration, the responses may overlap in phase and the total reaction be doubled, one must assume that $B(t)_{max} = 2A_0$. Then, to obtain the condition that $k_{max} < 1$, one must have the relationship

$$\frac{U_n}{U_s}\frac{\Delta\omega}{\omega_0} < 0.375. \qquad \text{(II.15)}$$

By taking into account that condition (II.15) must hold for $0 < \sqrt{\beta}\tau < 4$, we obtain the following formulas for the coefficients of expressions (II.10)-(II.12)

$$\nu_1 = 1, \quad \nu_2 = \frac{1}{\pi^2}\left(\frac{\Delta\omega}{\omega_0}\right)^2\left(\frac{U_n}{U_s}\right)^2 < 0.0143,$$

$$\nu_3 = \frac{2.38}{\pi^4}\left(\frac{\Delta\omega}{\omega_0}\right)^4\left(\frac{U_n}{U_s}\right)^4 \leqslant 0.000487.$$

For $\sqrt{\beta}\tau > 4$ and condition (II.14) being met,

$$\nu_1 = 1, \quad \nu_2 = 0.057, \quad \nu_3 = 0.00768.$$

From the values obtained for the ν, it is clear that, with an accuracy sufficient for our purposes, we can set

$$|\overline{F(\omega)}|^2 \approx |\overline{F_1(\omega)}|^2,$$

since

$$|\overline{F_1(\omega)}|^2 \gg |\overline{F_2(\omega)}|^2 \gg |\overline{F_3(\omega)}|^2.$$

Thus, the energy spectrum of the sequence of noise pulses at the discriminator output for $\omega_{max}/\Delta\omega = \omega_f/\Delta\omega \leq 0.2$ and $k_{max} = B(t)_{max} / U < 1$ is defined by the equation

$$W(\omega) \approx 2m\,|\overline{F_1(\omega)}|^2, \qquad \text{(II.16)}$$

where $|\overline{F_1(\omega)}|^2$ is found from formula (II.10)

The constant (dc) component of the sequence of noise pulses at the discriminator [7]

$$U_{\text{const}} = mF(0). \qquad (II.17)$$

equals zero, it following from (II.5) that $F(0) = A(0) - iB(0) = 0$.

Due to the absence of the constant component of (II.17), the discrete portion of the energy spectrum of the sequence of noise pulses at the discriminator output (consisting, in the general case, of a discrete and a continuous portion) will equal zero, and formula (II.1), defining the spectrum's continuous portion, will be valid, for equiprobable values of φ_0, both for chaotic and for periodic trains of noise pulses.

LITERATURE CITED

[1] R. D. Maurice, "V. H. F. broadcasting: Reduction of impulsive interference in f.m. reception," Electronic and Radio Engineer 34, new ser., 8 (1957).

[2] J. C. Tellier, "An analysis of the behavior of limiter-discriminator f. m. detectors in the presence of impulse noise," Proc. of National Electronics Conference (Chicago) 3 (1947).

[3] M. K. Zinn, "Transient response of an f.m. receiver," Bell System Techn. J. 27, 4 (1948).

[4] V. M. Sidorov, "The action of weak pulse noise on receivers of frequency modulated oscillations," Radiotekhnika 13, 3 (1958).

[5] L. B. Venchkovskii and Yu. I. Chugin, Actual Noise Stability of Frequency Receivers and the Statistical Characteristics of Noise, [in Russian] (Proc. IAT AN SSSR, 1957).

[6] S. I. Evtyanov, Transient Responses in Receiver-Amplifier Circuits [in Russian] (Svyaz'izdat, 1948).

[7] I. S. Gonorovskii, Radio Signals and Transient Phenomena in Radio Circuits [in Russian] (Svyaz'izdat, 1954).

ON AMPLIFIERS OF ERROR SIGNALS OF ELECTRICAL AUTOMATIC CONTROL SYSTEMS

L. L. Dekabrun

(Moscow)

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A new principle of designing synchronous detector circuits is presented which permits a significant lowering in the magnitude of the lag * introduced in the transmission of a signal by amplifiers of slowly varying voltage. The circuits permit the broadening of the domain of applicability of such amplifiers in automatic control systems where amplifiers with galvanic connections cannot provide high accuracy.

A qualitative analysis is given of the lag introduced by amplifiers in existing synchronous detection circuits.

The simplest amplifiers of error signals in automatic control systems are amplifiers with galvanic connections. Their disadvantage is that the drift of the output voltage [1, 2] and the indeterminacy of the initial state determine the limits of the accuracy with which the system can maintain a given value of the controlled quantity. Amplifiers with preliminary transformation of error signal voltages to ac carrier-frequency voltages [3] allow the system accuracy to be increased by at least two orders of magnitude and, with very good regulation, even higher, although in theory there is also drift in these amplifiers [4]. However, such amplifiers transmit signals with significant lag which, in some cases, can lead to a loss of system stability.

The "inertia" of amplifiers with transformed input voltages has received little treatment in the literature [5]. Below, we give an approximate quantitative estimate of this quantity, and present a significantly less "inertial" principle for designing such amplifiers.

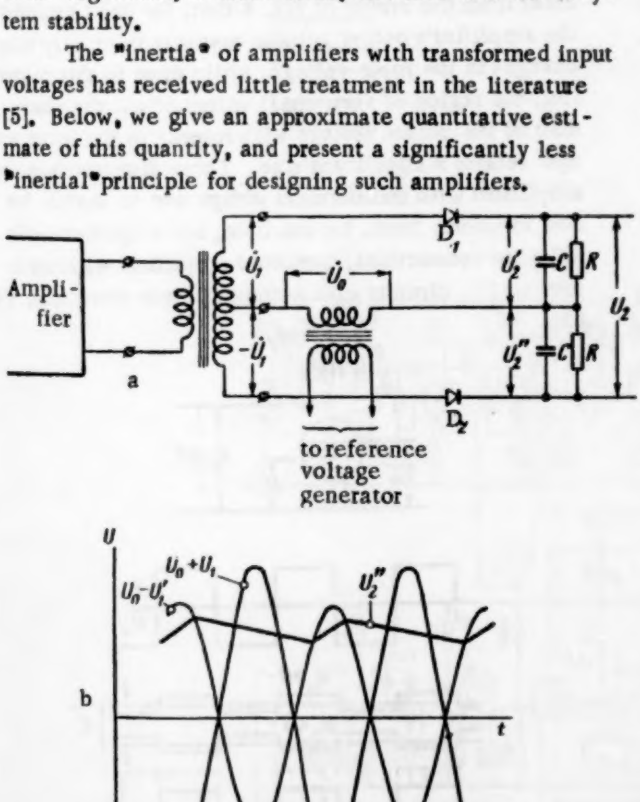

Fig. 1.

The lag in amplifiers with transformed input voltage is induced by the filters which smooth the voltage at the output of the synchronous detector circuit following the amplifiers. Figure 1 a shows one of the widely used synchronous detector circuits; Fig. 1 b shows a somewhat simplified graph of the output voltages U_2' and U_2'' of each section of this circuit. The difference of these voltages constitutes the output voltage U_2 of the synchronous detector and amplifier as a whole.

For adequate smoothing of the output voltage, it is necessary that the following condition hold

$$T_0 \ll \tau, \tag{1}$$

where T_0 is the period of the carrier frequency voltage off which the transformer of the input voltage operates and τ = RC is the smoothing filter's time constant. When condition (1) holds, the dc component of voltage U_2 will have the magnitude

$$U_{20} = 2U_1\left(1 - \frac{T_0}{2\tau}\right), \tag{2}$$

and the relative magnitude of the pulsation is found from the formula

$$\varepsilon = \frac{(U_{\text{pulse}})_{\max}}{U_{20}} = \frac{\frac{T_0}{\tau}}{2\left(1 - \frac{T_0}{2\tau}\right)}. \tag{3}$$

Figure 2 shows U_{20} and ϵ as functions of the smoothing filter's time constant.

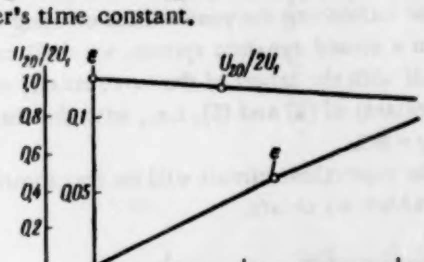

Fig. 2.

* In this paper, we understand by the terms "lag" and "inertia" a sum of factors which give rise to any distortion, other than scale-factor changes, of the output signal as compared to the input signal.

The transient response of the synchronous detector (and, consequently, of the amplifier as a whole), i.e., its reaction to an abrupt change in amplitude of the detected voltage U_1, is a complicated function of the circuit parameters and of time. By simplifying the outlines, we can show that

a) for an abrupt decrease of U_1, the fall of the output voltage will follow a law close to an exponential one with a time constant somewhat larger than

$$\tau_f = RC; \tag{4}$$

b) for an abrupt increase in U_1, the rise of the output voltage will follow a law close to an exponential one with a time constant somewhat larger than

$$\tau_r = \frac{RR_1}{R + R_1} C. \tag{5}$$

Here, R_1 is the resulting resistance of the circuit through which the condenser charges (the resistance of the diode, the transformer, etc.).

The rise and fall of the output voltage do not follow purely exponential laws because, in the given circuit, they are not continuous processes, e.g., with rising voltage, the condensers only charge during fractions of the positive half-periods while, during the rest of the time, they discharge through their load resistances.

We now construct the amplitude and phase frequency characteristics of the amplifier, with the assumption that its input voltage, i.e., the error signal voltage, varies harmonically

$$U_{in} = U_м \cos \omega_1 t. \tag{6}$$

For a given frequency ω_0 of the modulating voltage, the frequency ω_1 of the input voltage must lie within the limits of

$$0 \leqslant \omega_1 \leqslant (0.1 \div 0.2)\omega_0, \tag{7}$$

since otherwise the amplifiers's modulating device could not transmit the form of the input voltage with sufficient accuracy [3]. With this, the detected voltage U_1 will be a sinusoid with variable amplitude

$$U_1 = (U_м \cos \omega_1 t) \sin \omega_0 t. \tag{8}$$

For estimating the possibilities of using the amplifier in a closed dynamic system, we shall consider its circuit with the larger of the two actually existing time constants of (4) and (5), i.e., with the time constant τ_f = RC.

The equivalent circuit will be that shown on Fig. 3, from which we obtain

$$\frac{U_{20}}{kU_{in}} = \frac{1}{\sqrt{1 + (\omega_1 \tau_f)^2}}, \tag{9}$$

$$\operatorname{tg} \psi = -\omega_1 \tau_f. \tag{10}$$

Here, k is the over-all gain and ψ is the phase shift of the amplifier's output voltage with respect to its input voltage.

Figure 4 shows the amplitude and phase frequency characteristics of the amplifiers computed by these formulas. The parameter of the characteristics is the relative time constant of the detector's smoothing circuit, $\beta = T_0/\tau$. These characteristics are universal, being valid for all forms of synchronous detector circuits, in-

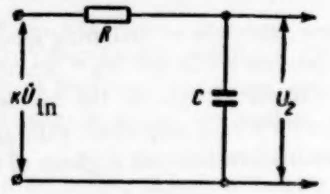

Fig. 3.

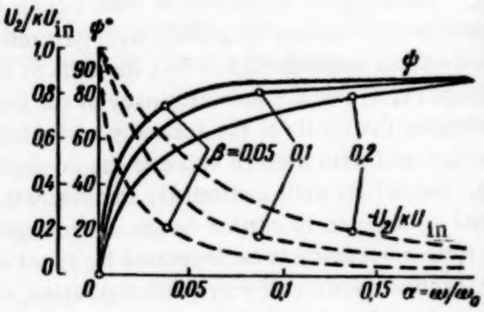

Fig. 4.

cluding the so-called fast-acting variants of [6]. It is clear from the curves of Fig. 4 that, for good smoothing, the amplifier's output voltage reacts only to very slow changes in the input voltage, while even in this region, i.e., the region of very small values of ω_1, the phase shift of the output voltage with respect to the input voltage retains a significant size. These disadvantages of amplifiers with transformers oblige one to search for new solutions. Such, for example, are amplifiers with galvanic connections, corrected amplifiers with transformers [7], circuits with automatic zero correction [8], etc.

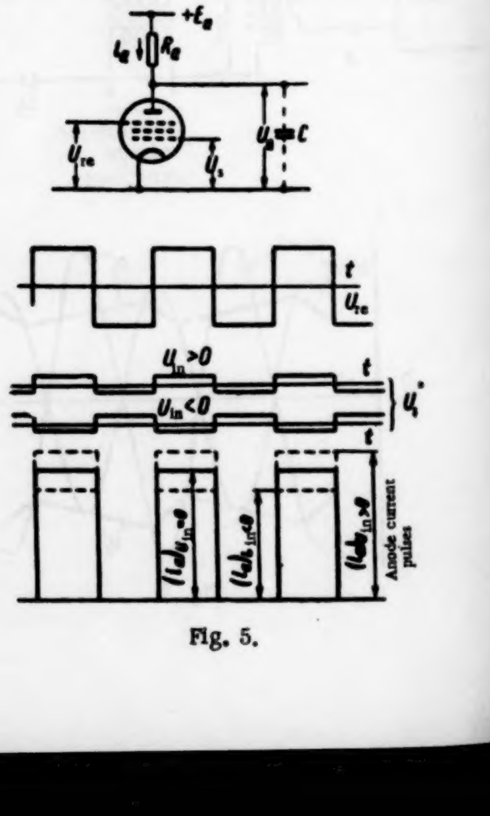

Fig. 5.

The "inertia" of an amplifier with transformation of the input voltage can be significantly decreased if the smoothing of the output voltage is implemented, not by means of accumulators of energy (on the circuit of Fig. 1, the capacitor C is such an accumulator of energy), but by the filling of the "holes" between the pulses of output voltage by pulses obtained from a detection circuit. If the transforming device transforms the input voltage to rectangular pulses (as, for example, in the case of transformation by means of a contact chopper) and the reference voltage also has the form of rectangles, one could then theoretically get rid of the smoothing condensers at the output. Figure 5 shows the block schematic of a synchronous detector and the graphs characterizing its operation in the case when the reference and the detected voltages have the form of rectangular pulses. We assume that there is a second such circuit, operating on the same load resistance R_a but at those intervals of time when the first circuit is cut off by the reference voltage, i.e., the operating cycle of the second circuit is shifted by π with respect to the operating cycle of the first circuit. Then, in the ideal case, the output voltage (the voltage on load R_a) will have no pulsations. Figure 6 gives the functional block schematic of an amplifier with a transformer of the input voltage and an inertialess two-cycle synchronous detector, while Fig. 7 gives the graphs which characterize its operation. The difficul-

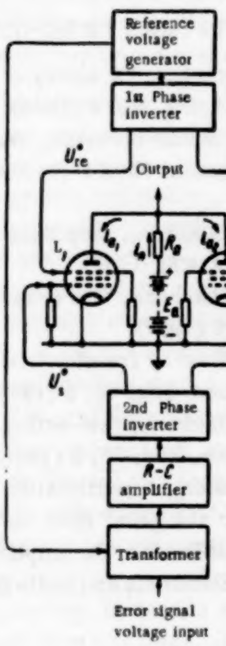

Fig. 6.

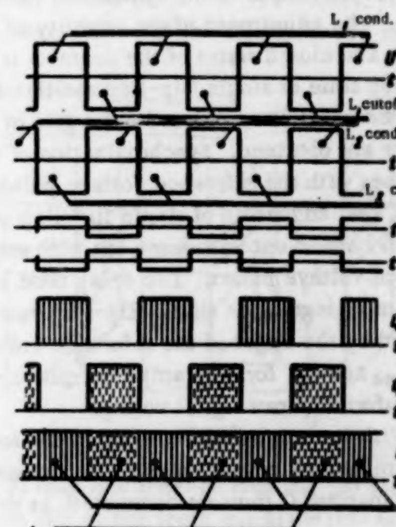

Fig. 7.

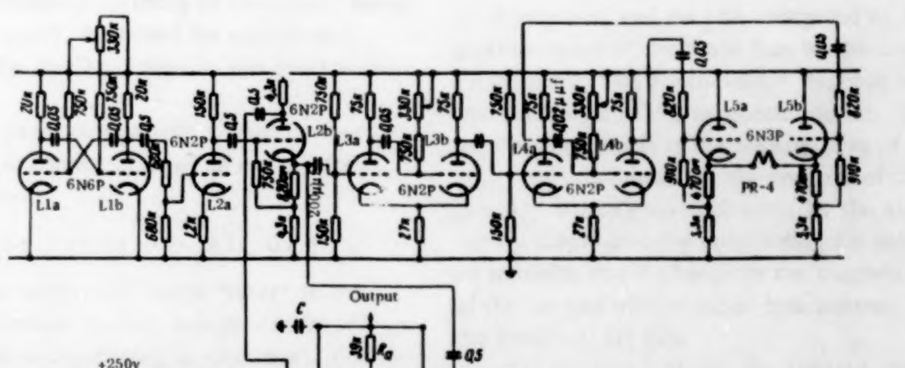

Fig. 8.

ties related to the realization of this principle have to do, principally, with the accuracy of synchronization of the operating cycles, and with the accuracy of matching the characteristics of the two channels. Figure 8 shows a circuit which is one of the possible variants of the solution to this problem. The basis of the circuit is a two-cycle synchronous detector (tubes L_7 and L_8) analogous to that considered above. The circuit's rhythm of operation is given by the independently driven multivibrator $L_{1a}L_{1b}$ in whose circuit there is the capability of tuning symmetric cycles. The reference voltage of the synchronous detection circuit is formed by amplifier-limiter L_{2a} and split by phase-inverter L_{2b}. The contact transformer of the error signal voltage, implemented by polarized relay PR-4, is connected at the output of the symmetric cathode follower $L_{5a}L_{5b}$. The adjustment of the equality of the intervals of open and closed states of the contacts is regulated by the delay time of single flip-flop oscillator $L_{4a}L_{4b}$, from which the controlling pulses for the grid of the cathode follower are obtained. Synchronization of the input voltage pulses with the reference voltage pulses is so performed, that triggering of single flip-flop oscillator $L_{4a}L_{4b}$ is carried out with some lag with respect to the reference voltage pulses. The delay time is established by means of regulation single flip-flop oscillator $L_{3a}L_{3b}$, triggered by the edges of the reference voltage pulses. Tubes L_{6a} and L_{6b} form an amplifier-phase-inverter for the transformed error signal voltage.

With inexact adjustment, there are some pulsations in the amplifier's output voltage, for the smoothing of which capacitor C must be connected, as shown with a dashed line on the circuit diagram. However, even with comparatively inexact adjustment, to attain smoothing of the output voltage in the given circuit which is comparable with the smoothing obtained in the ordinary synchronous detector circuits, there is required a time constant of the smoothing circuit which is less than the usual one by several tens.

The gain of the system shown on Fig. 8 is about 400. To increase the gain, it is necessary to include one or more amplifying stages before the circuit of phase-inverter $L_{6a}L_{6b}$.

LITERATURE CITED

[1] C. Verhagen, "A survey of the limits in dc amplification," Proc. IRE No. 5 (1953).

[2] M. A. Bonch-Bruevich, The Use of Electronic Tubes in Experimental Physics [in Russian] (Gostekhizdat, 1955).

[3] L. L. Dekabrun, Amplifiers and Voltage Supplies [in Russian] (MIFI, 1958).

[4] S. C. Hutcheon, "Dc to ac converter circuits," Proc. IRE 104C, 6 (1957).

[5] F. H. Krentz, "Transfer function of dc chopper amplifier," Electron. Eng. 76, 9 (1957).

[6] D. L. Davis, "A fast-acting phase-conscious indicator," Electron. Eng. 75, 9 (1956).

[7] E. Goldberg, "Stabilization of wide-band dc amplifier for zero and gain," RCA Review 11, 20 (1950).

[8] M.N. McFadden, "Dc amplifier with zero input correction," Electronic and Radio Engineering 10, 10 (1957).

DETERMINATION OF MAGNETIC CONDUCTANCE WITH GEARED STATOR AND ROTOR

L. A. Sribner

(Odessa)

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A method is considered for the calculation of magnetic conductance for a geared stator and rotor, with flux leakage being taken into account. Functional relationships are established for the determination both of the total magnetic conductance and of the coefficients which take into account flux leakage.

The method developed can be used for designing electric hammers, geared inductive transducers, pitch and pulsed motors, and other electromagnetic systems, and also for the computation of the flux pulsation in the teeth of electrical machines with geared rotors and stators.

In contemporary technology, broad use is made of various electromagnetic mechanisms with moving and immovable geared portions.

The computation of the magnetic circuits of such mechanisms is virtually untreated in the literature. Ordinarily, for computing the magnetic conductance of an air gap, it is recommended that one use awkward graphical methods. The greatest difficulty is presented by the accounting for flux leakages. If one knows the magnetic conductance with flux leakages taken into account, the problem is significantly simplified. An extensive literature [1-3] is devoted to computational methods for this case.

The flux leakage in certain mechanisms with geared rotors and stators is large, and sometimes exceeds significantly the basic flux. In some of these mechanisms the flux leakage is almost constant [4], but it varies sharply in the overwhelming majority of the cases. Sometimes, Carter's formula [5, 6] is used for calculating leakage fluxes, but for small air gaps its use gives understated results.

In the present paper we present a formula for calculating the magnetic conductance of an air gap for geared stators and rotors.

Definition of the Fictitious Air Gap

We shall agree to give the name "rotor" to the moving part of a magnetic circuit, independently of whether its motion is reciprocating or rotational. The nonmoving part of the circuit we call the stator. The flux between the teeth of the stator and rotor we shall call the basic magnetic flux. The magnetic fluxes in the grooves, in the face ends, and through the ribs of the teeth we shall call the leakage fluxes. It must be said at once that, in many cases (for example, in electrical machines), the leakage flux is not a parasitic flux, but is added to the useful basic flux. However, even in this case, we shall call it a leakage flux.

There are the following leakage fluxes between the stator and rotor teeth (Fig. 1): 1) lateral leakage

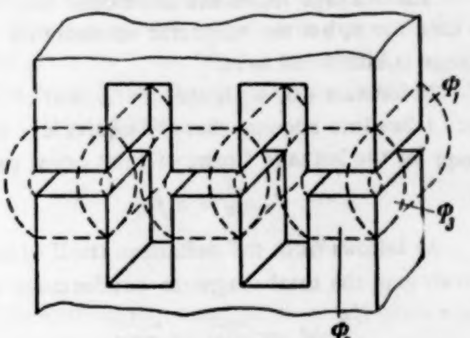

Fig. 1.

flux Φ_1; 2) face end leakage flux Φ_2; 3) leakage flux through the ribs of the teeth Φ_3; 4) the leakage flux of the coil, i.e., the leakage flux through the air around the coil.

The leakage flux of the coil is constant for practical purposes, and may be computed by multiplying the greatest value of the basic flux by the coefficient $\sigma = 0.05$ to 0.2. The coefficient σ depends on the design characteristics of the magnetic circuit. In the sequel, we shall not consider the leakage flux of the coil.

We now introduce the concept of the fictitious air gap δ'. For this we shall consider the air gap to be uniformly distributed (by considering the rotor and stator to be smooth), and a change in the magnetic conductance of the air gap will be taken into account by a change in the fictitious air gap.

If the flux leakages are ignored, then the fictitious air gap will equal

$$\delta' = \frac{\delta_0}{\frac{a}{t}\left(1 - \frac{l}{a}\right)} = \frac{\delta_0}{0.5\left(1 - \frac{l}{a}\right)}, \qquad (1)$$

where δ_0 is the actual air gap, a is the width of the teeth, $t = 2a$ is the pitch of the teeth, i.e., the total width of the tooth plus the groove, and l is the relative shift of the rotor and stator teeth.

We now consider how one may take leakage fluxes into account with the determination of the fictitious air

gap δ'. For simplicity of the reasoning, we shall assume that the rotor teeth are against the stator teeth, i.e., that $l = 0$.

The magnetic conductance of the basic flux on unit tooth length equals

$$g = \frac{\mu_0 S}{\delta_0} = \frac{\mu_0 a}{\delta_0}, \tag{2}$$

where μ_0 is the magnetic permeability of air and S is the area through which the basic flux passes. In our case, S = a.

Moreover, the flux leakages induce additional magnetic conductance. This conductance, per unit tooth length, equals

$$g_1 = k_1 \frac{\mu_0 2a}{\delta_0}. \tag{3}$$

The leakage fluxes are not closed only through the area S = a, but the magnetic conductance of the leakage is attributed to it.

In formula (3), k_1 is the coefficient of leakage which takes into account that the equivalent air gap $\delta_{0\,equ}$ for the leakage fluxes will not equal gap δ_0:

$$\delta_{0\,equ} = \delta_0 / k_1.$$

As follows from the definition itself of the fictitious air gap, the total magnetic conductance equals

$$g_t = \frac{\mu_0 2S}{\delta'} = \frac{\mu_0 2a}{\delta'}. \tag{4}$$

On the other hand, the total magnetic conductance is

$$g_t = g + g_1.$$

By substituting the values of the conductances from formulas (2), (3), and (4), we get that

$$\frac{\mu_0 2a}{\delta'} = \frac{\mu_0 a}{\delta_0} + k_1 \frac{\mu_0 2a}{\delta_0} \tag{5}$$

or, after some transformations,

$$\delta' = \frac{\delta_0}{0.5 + k_1}.$$

If the rotor teeth are shifted relative to the stator teeth, i.e., if $l \neq 0$, then, by taking formula (1) into account, we get

$$\delta' = \frac{\delta_0}{0.5\left(1 - \frac{l}{a}\right) + k_1}. \tag{6}$$

Taking into Account the Magnetic Conductance of the Lateral Flux Leakage

The coefficient which takes into account the magnetic conductance of the lateral (side) flux leakage depends, in addition to everything else, on the shift of the rotor teeth relative to the stator teeth. We shall therefore first determine it for the case when the rotor teeth are against the stator teeth.

It is clear from expression (3) that the computation of the coefficient of side leakage $k_s = k_1$ reduces to the computation of the magnetic conductance of the lateral leakage per unit length $g_s = g_1$, since the remaining quantities of expression (3) are already known.

If we assume that the magnetic lines of forces are semicircles, then the conductance equals [2]

$$g_1 = \frac{\mu_0}{\pi} \ln\left[1 + \frac{2a}{\delta_0}\right]. \tag{7}$$

If they are assumed to be semi-ellipses, then [2]

$$g_1 = \frac{\mu_0}{\pi} \ln\left[1 + 2\frac{a + \sqrt{a^2 + a\delta_0}}{\delta_0}\right]. \tag{8}$$

If they are considered to consist of four portions of circles and straight lines, we then obtain [2]

$$g_1 = \frac{\mu_0}{\pi} \ln\left[1 + \frac{\pi a}{\delta_0}\right]. \tag{9}$$

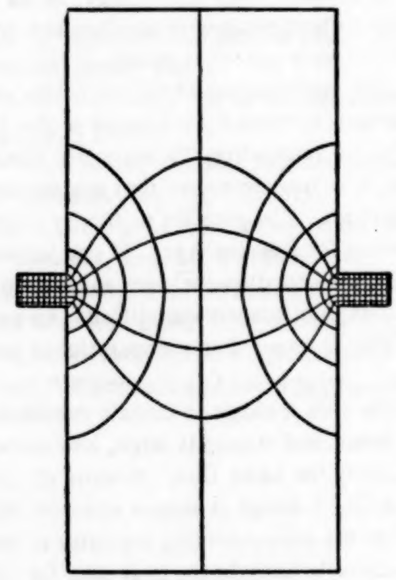

Fig. 2.

We now consider what form the actual lines of force might have. On Fig. 2, we have constructed the outlines of the magnetic field in the grooves for the groove depth equal to its width. For this construction, the steel surface was taken as an equipotential surface which, in the given case, is completely admissible, even for comparatively large values of induction.

It is clear from Fig. 2 that the magnetic lines of force differ significantly from semicircles or parts of ellipses. Moreover, for various values* of x, the magnetic lines of force are described by different equations and, for x = a, reduce to straight lines. In this case, the mean length of the magnetic lines of force equals approximately $l_x = 3x + \delta_0$; for small x the length is $l_x > 3x + \delta_0$ and, for large x, $l_x < 3x + \delta_0$ and, for increasing x, approximates to the value $l_x = 2x + \delta_0$.

*We denote by x the distance between the magnetic lines of force and the beginning of the groove for equipotential surfaces equally spaced from the teeth.

The area through which the magnetic flux passes is not constant, but varies per unit length from 0.5 to 1.5a, there being a corresponding change in the area of the elementary force tubes from 0.5 to 1.5 dx; in the mean, this area equals dx.

The estimates given are also approximately valid for other relationships of the size of the air gap and the teeth dimensions.

By substituting these mean values of length and area of the elementary force tubes in the formula for computing magnetic conductance, we get

$$g_s = g_1 = \mu_0 \int_0^a \frac{dx}{3x + \delta_0} =$$

$$= \frac{\mu_0}{3} \left| \ln (3x + \delta_0) \right|_0^a = \frac{\mu_0}{3} \ln \left(1 + \frac{3a}{\delta_0}\right). \qquad (10)$$

Finally, we take the following formula which gives the results coinciding most exactly with experimental results:

$$g_s = \frac{\mu_0}{\pi} \ln \left(1 + \frac{3a}{\delta_0}\right). \qquad (11)$$

From expressions (3) and (11), we obtain the expression for the coefficient of lateral (side) leakage

$$k_s = \frac{\delta_0 \ln \left(1 + \frac{3a}{\delta_0}\right)}{\pi a}. \qquad (12)$$

We now consider how the coefficient of lateral conductance varies as the rotor teeth are shifted (displaced) with respect to the stator teeth.

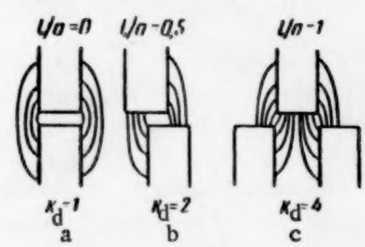

Fig. 3.

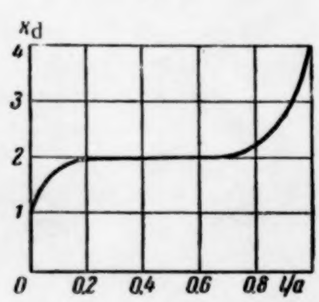

Fig. 4.

Let the rotor teeth be displaced relative to the stator teeth by half a tooth (Fig. 3 b). In this case the conductance $g_1 = \mu_0 \int_0^a dx/l_x$ will be twice as large as in the case with nondisplaced teeth (Fig. 3 a), since here the length of the magnetic lines of force is twice as small. Consequently, the coefficient of side conductance will be twice as large here. The coefficient of side (lateral) conductance will be exactly the same if there are teeth only on the stator, the rotor being smooth, or conversely.

We now consider the case when the rotor teeth are against (opposite) the stator grooves (Fig. 3 c). Here, the coefficient of lateral conductance will be twice as large as in the previous case, i.e., four times larger than for nondisplaced teeth. We take the variation of the coefficient of lateral displacement into account by introducing a coefficient of teeth displacement k_d, which is defined by the graph of Fig. 4. This graph was constructed with experimental results being taken into account, these results showing that the curve of the magnetic flux of such electromagnetic mechanisms is a straight line when the teeth displacement lies within the limits of $l / a = 0.2$ to 0.7 and, for larger and for smaller values of l / a, becomes curved. Analogous curves of the flux were also given by Yasse [4] (for two teeth).

The formula for determining the fictitious air gap takes the following form:

$$\delta' = \frac{\delta_0}{0.5\left(1 - \frac{l}{a}\right) + k_d k_s}. \qquad (13)$$

This formula can be applied in all cases when there are narrow and long grooves. If the grooves are only on the rotor, or only on the stator, then the coefficient k_s must be doubled, whereby it no longer depends on the position of the rotor. Then, the formula is written in the following form:

$$\delta' = \frac{\delta_0}{0.5 + k_s}. \qquad (14)$$

For electrical machines, formulas (13) and (14) give results which coincide closely with those computed by Carter's formula [7].

Further Refinement of the Formula for Determining the Fictitious Air Gap

Formula (13) gives incorrect results for short grooves. In this case it is necessary to take into account the face leakage flux and the leakage flux through the ribs of the teeth.

The coefficient of face end leakage can be computed analogously to the coefficient of lateral leakage, assuming the depth of the grooves to be significantly greater than the width, but the active depth equal to the groove width.

The length of the magnetic lines of force can be taken equal to $l_x = 3.5x \pm \delta_0$, on the average, and the cross section of an elementary magnetic force tube per unit length equal, on the average, to approximately dx.

The magnetic conductance of the face end flux leakage equals

$$g_f = \mu_0 \int_0^a \frac{dx}{l_x} = \mu_0 \int_0^a \frac{dx}{3.5\,x + \delta_0} =$$

$$= \frac{\mu_0}{3.5}\left|\ln(3.5x + \delta_0)\right|_0^a = \frac{\mu_0}{3.5}\ln\left(1 + 3.5\,\frac{a}{\delta_0}\right), \quad (15)$$

from whence we obtain, for the coefficient of face end conductance,

$$k_f = \frac{\delta_0 \ln\left(1 + 3.5\,\frac{a}{\delta_0}\right)}{3.5a}\,\frac{a}{b} = \frac{\delta_0 \ln\left(1 + 3.5\,\frac{a}{\delta_0}\right)}{3.5b}, \quad (16)$$

where b equals the groove length.

As the rotor teeth are displaced relative to the stator teeth, the coefficient of face end conductance, i.e., the leakage flux through the face end, just as with the basic flux, decreases linearly.

We now take into account the conductance of the teeth ridges. If one equates the conductance of the ridges with the conductance of the face end, it is then easily calculated that the coefficient of ridge flux for undisplaced teeth equals $0.6k_f$. When the rotor teeth are opposite the stator grooves, this coefficient is doubled, since the flux contains an angle twice as large. We assume that the coefficient of flux leakage through a ridge varies linearly as the rotor teeth as shifted with respect to the stator teeth (although, in actuality, it varies in accordance with a more complicated law).

The formula for calculating the fictitious air gap when one takes into account the coefficients of face end flux leakage and ridge flux leakage takes the following form:

$$\delta' = \frac{\delta_0}{\left(1 - \frac{l}{a}\right)(0.5 + k_f) + 0.6k_f\left(1 + \frac{l}{a}\right) + k_d k_s}, \quad (17)$$

where k_s and k_f are defined by formulas (12) and (16).

Taking the Reluctance of the Magnetic Circuit into Account

Formula (17) makes it possible to calculate the magnetic conductance of an air gap with the flux leakages taken into account. However, for small air gaps δ_0, the reluctance of the magnetic circuit becomes commensurable with the reluctance of the air gap, and must therefore be taken into account.

As is well known, for ferromagnetic materials the dependence on ampere-turns of the magnetic flux is nonlinear, and is given by the magnetization curve. Consequently, the magnetic circuit's reluctance is not constant, but depends on the ampere-turns applied.

It is simplest to take the magnetic circuit's reluctance into account graphically. For this, we decompose the magnetic circuit into a series of portions (segments) such that the cross-section of the magnetic circuit will be invariant on each segment. For each portion one constructs, from the magnetization curve for the given type of steel, the magnetization curve of the segment multiplied by the value of induction from the first curve on the segment's cross section, and the field strength (specific ampere-turns) on the segment's length.

Thereafter, one constructs the magnetization curve of the entire magnetic circuit, adding up, for arbitrary values of magnetic flux, the ampere-turns of the segments of the magnetic circuit.

After this, it is necessary to construct the dependence of the magnetic flux on the ampere-turns for the air gap. For this, we give ourselves the ampere-turns iw and compute the magnetic flux Φ_0 in the air gap under the assumption that the magnetic circuit's reluctance equals zero:

$$\Phi_0 = \frac{iw}{\delta'}\,\mu_0 S, \quad (18)$$

where S is the area of the air gap.

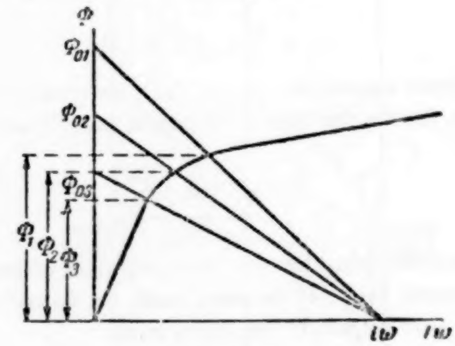

Fig. 5.

Since the magnetic flux depends linearly on the ampere-turns for the air gap, the corresponding graph can be constructed from two points by drawing the straight line which passes through the origin and through the point with coordinates iw, Φ_0. For convenience, we construct this line by the points with coordinates iw, 0, and 0, Φ_0 (Fig. 5).

Then, the point of intersection of the magnetic circuit's magnetization curve and the line for the air gap will also be the graphical solution of the problem, i.e., we shall have found the magnetic flux of the magnetic system Φ for the given ampere-turns. If the flux is divided by the ampere-turns, the quotient will be the magnetic conductance for the given conditions.

Such computations can be repeated for various shifts of the rotor teeth relative to the stator teeth, i.e., for various fictitious gaps. Naturally, the magnetization curve of the magnetic circuit will not thereby be varied.

The data thus obtained can be employed in the design of all the possible electromagnetic devices.

The magnitude of the magnetic conductance and its variations as a function of the displacement of the rotor teeth relative to the stator teeth can be converted to terms of forces, moments (torques), or inductions, depending on the purpose for which the device being designed is intended.

In conclusion, the author wishes to thank L. K. Shrago for reading the manuscript and for making a number of valuable comments.

LITERATURE CITED

[1] A. Ya. Builov, Fundamentals of Electrical Apparatus Construction [in Russian] (Gosénergoizdat, 1946).

[2] B. S. Sotskov, Elements of Automation and Remote Control Apparatus [in Russian] (Gosénergoizdat, 1950).

[3] G. K. Roters, Electromagnetic Mechanisms [in Russian] (Gosénergoizdat, 1949).

[4] E. Yasse, Electromagnets [in Russian] (Gosénergoizdat, 1934).

[5] A. I. Moskvitin, "On the conductance of a gap with opened grooves on stator and rotor," Izvest. Akad. Nauk SSSR, Otd. Tekh. N. 1 (1941).

[6] A. I. Moskvitin, Reciprocating Electrical Machines [in Russian] (Izd. AN SSSR, 1950).

[7] Carter, Air gap induction, El. World and Eng., 38 (1901).

THE EFFECT OF NONUNIFORMITY OF MAGNETIZATION ON THE STATIC CHARACTERISTICS OF CORES. I

G. D. Kozlov

(Moscow)

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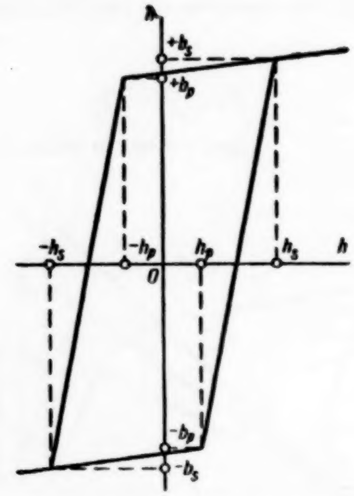
The effect of nonuniformity of magnetization on the static characteristics of cores is considered. The computations are given for the limiting magnetization polarity reversal loop of a core, the symmetric and biased magnetization polarity reversal loops, and the "basic magnetization curve".

Recently, in various branches of automation, remote control, and computing technology, wide use has been made of magnetic amplifiers employing cores with complicated forms of magnetic circuits (magnetic circuits with variable cross sections, ramified or multibranched magnetic circuits, etc.). Among the magnetic devices with ramified magnetic circuits are the "transfluxers" which can be used as decoders, logical elements, memory devices without destructive read-out, etc. The magnetization of the magnetic circuit in such devices occurs nonuniformly, and is fundamentally conditioned by the geometric form and the dimensions of the magnetic circuit, and also by the physical position of the magnetizing windings. In [1, 2] was considered the question of taking into account the effect of nonuniform core magnetization on its static characteristics, the present work being a development of the earlier ones. In [1, 2], the authors determine the magnitude of the magnetic induction of a toroidal core as a function of the field strength, B = f(H). With this, the field strength, is taken, in [1], relative to the mean core radius $R = (R_o + R_i)/2$ and, in [2], relative to the mean harmonic radius $R_h = (R_o - R_i / \ln(R_o/R_i)$.* From our point of view, one can more simply and more graphically derive conclusions by using the relationship B = f(F) instead of B = f(H), where F is the number of ampere-turns of the magnetization. The advantage of introducing such a relationship is particularly clearly seen when one takes into account the nonuniformity of magnetization in a multibranched magnetic circuit, where the relationship B = f(H) loses its meaning, since it is not known relative to which length of lines of force the field strengths are defined.

The present work is devoted to accounting for the effect of nonuniformity of core magnetization on its static characteristics for various core geometries and for various modes of magnetic polarity reversal. In this portion of the work, all the basic relationships are derived for toroidal magnetic circuits when they undergo complete, and partial, reversal of their magnetic polarity.

Basic Statements

The hysteresis loops of all magnetic material can, with sufficient accuracy, be presented by two forms of



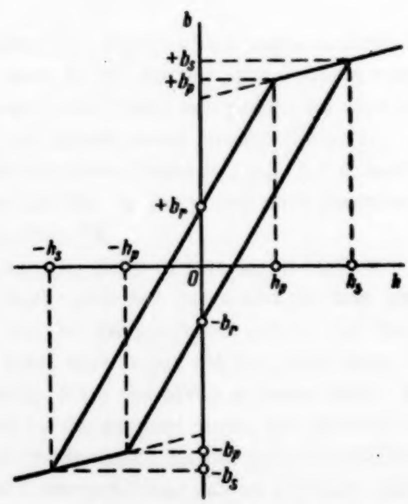


Fig. 2.

idealized hysteresis loops (Figs. 1 and 2). Each form of loop corresponds to a definite group of materials. Within each group, the materials can be distinguished from each other by the following parameters; b_s, the saturation induction; b_p, the induction at the beginning of magnetic polarity reversal; h_s the saturation field

* Publishers' note – "o" and "i" are "outside diameter" and "inside diameter," respectively.

strength; h_p the field strength at the beginning of magnetic polarity reversal. Such idealized material hysteresis loops permit one to determine, not only qualitatively, but also quantitatively, the effect of nonuniformity of core magnetization on its static characteristics. The process of magnetic polarity reversal of a core presents itself as the reversal of magnetic polarity of individual concentric layers of material lying one within the other. The reversal of magnetic polarity begins with the inner levels, when the induced field strength becomes greater than the quantity h_p. The reversal of magnetic polarity ceases when the magnitude of the field strength reaches the value h_s.

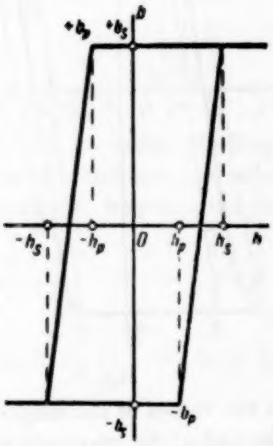


Fig. 3.

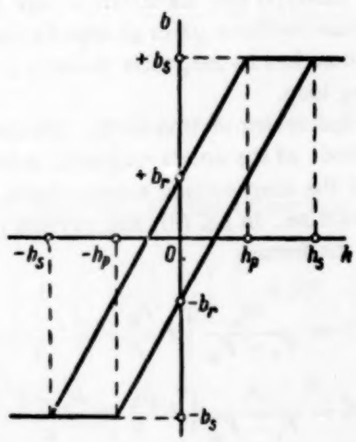


Fig. 4.

For simplicity, we shall consider the material hysteresis loops shown on Figs. 3 and 4, for which the ascending and descending arms of the hysteresis loops are expressed mathematically as follows:

For the case of Fig. 3,

$$b = \frac{2b_s}{h_s - h_p}\left(h - \frac{h_s + h_p}{2}\right), \tag{1}$$

$$b = \frac{2b_s}{h_s - h_p}\left(h + \frac{h_s + h_p}{2}\right); \tag{2}$$

for the case of Fig. 4,

$$b = \frac{2b_s}{h_s + h_p}\left(h - \frac{h_s - h_p}{2}\right), \tag{3}$$

$$b = \frac{2b_s}{h_s + h_p}\left(h + \frac{h_s - h_p}{2}\right). \tag{4}$$

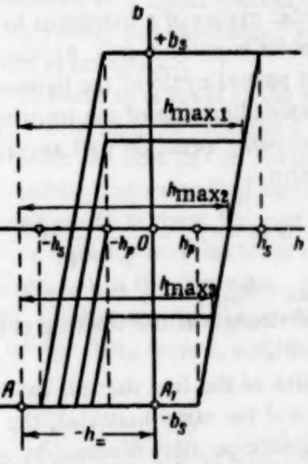


Fig. 5. Family of symmetric cycles of a hysteresis loop.

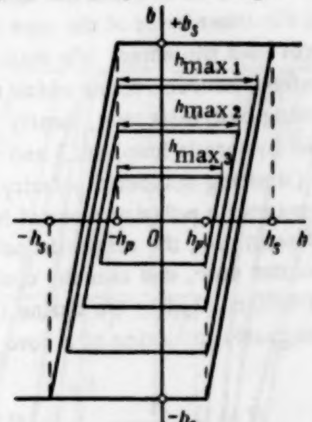


Fig. 6. Family of biased cycles of a hysteresis loop.

For the consideration of the process of reversal of a material's magnetic polarity, we shall henceforth assume that the ascending and descending arms are symmetric and have the same slope in the partial hysteresis loops as the arms of the limiting hysteresis loop. This assertion is borne out to a significant degree by the experimental data provided in the literature. We shall therefore assume that the characteristic slope of the material, $b = f(h)$, is constant for the ascending and descending arms, and equal to $2b_s/(h_s - h_p)$ (for the case of Fig. 3) independently of the form of the hysteresis loop by which the process of magnetic polarity reversal occurs. The quantity h_p remains constant both for symmetric and for biased partial cycles of a hysteresis loop (Figs. 5 and 6). This permits the ascending and descending arms of the family of symmetric cycles of a hysteresis loop (Fig. 5) to be described by the following equations:

$$b = \frac{2b_s}{h_s - h_p}\left(h - \frac{h_{max} + h_p}{2}\right), \tag{5}$$

$$b = \frac{2b_s}{h_s - h_p}\left(h + \frac{h_{max} + h_p}{2}\right). \qquad (6)$$

Here, h_{max} is the maximum value of the magnetizing field strength.

In reversing a material's magnetic polarity by biased partial cycles of a hysteresis loop, when the initial working point is point A (Fig. 6), the ascending arm of the biased partial cycle of the hysteresis loop coincides with the ascending arm of the limiting hysteresis loop, but the following equation will actually hold for the descending arm

$$b = -b_s - \frac{2b_s}{h_s - h_p}(h' - h_{max}'), \qquad (7)$$

where $h'_{max} = h_{max} - h_{=}$ and $h' = h - h_{=}$ is the value of the field strength if the working point is translated to A_1 (Fig. 6).

In spite of the fact that all the layers of a core are of one and the same material, the core as a whole has its magnetic polarity reversed by a loop which differs from the material's hysteresis loop. The explanation of this is that, for one and the same number of ampere-turns, the inner layer of the core has a greater field strength than the outer. We shall therefore distinguish henceforth between terms which relates to the material (limiting hysteresis loop, family of symmetric cycles of the hysteresis loop, etc.) and terms referring to the core (limiting magnetic polarity reversal loop, symmetric magnetic polarity reversal loop family, etc.).

For determining the magnetic polarity reversal loop of an entire core, one uses the concept of the mean magnetic induction [1, 2]. We define the magnitude of the mean magnetic induction of a core by the equation

$$B = \frac{1}{R_o - R_i}\int_{R_i}^{R_o} b\,dR, \qquad (8)$$

where R_o and R_i are the core's outer and inner radius and $\alpha = R_o / R_i$ is the ratio of the radii.

To find the relationship $B = f(F)$, we substitute the transformed Eqs. (1)-(7) in Eq. (8).

The Core's Magnetic Characteristics

Depending on the maximum number of magnetizing ampere-turns F_{max}, the core will have different modes of magnetic polarity reversal, defining the form of the function $B = f(F)$.

We initially consider the operation of a core in the mode of its limiting magnetic polarity reversal loop, if the materials's hysteresis loop has the form shown on Fig. 3.

a) Magnetic Polarity Reversal by the Limiting Loop ($F_{max} \geq \alpha F_s$)

Figure 7 shows the construction of the core's limiting magnetic polarity reversal loop in terms of the coordinates B and F. We denote by the lines aa_1 and cc_1 the ascending and descending arms of the hysteresis loops for the inner and outer layers of the material, respectively. The values of F_p and F_s define the ampere-turns of the inner layer's magnetic polarity reversal and saturation, respectively. The values of αF_p and αF_s serve similarly for the outer layer. In considering Fig. 7, one

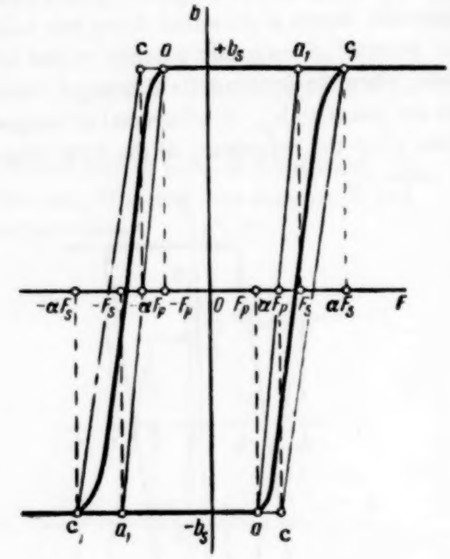


Fig. 7.

may note that the values of the ampere-turns for the beginning and the end of the magnetic polarity reversal for different core layers do not coincide. For a given magnetic material and for different core geometries, one can consider three cases of core induction variation when the core has its magnetic polarity reversed along its limiting loop.

For the determination of the function $B = f(F)$, i.e., the magnitude of the core's magnetic induction as a function of the magnetizing ampere-turns, it is necessary to substitute, in Eq. (8), Eqs. (1) and (2), transformed to the form:

$$b = \frac{2b_s}{F_s - F_p}\left(F\frac{R_i}{R} - \frac{F_s + F_p}{2}\right), \qquad (9)$$

$$b = \frac{2b_s}{F_s - F_p}\left(F\frac{R_i}{R} + \frac{F_s + F_p}{2}\right). \qquad (10)$$

In the first case (Fig. 7), the core's geometry is such that $\alpha F_p < F_s$, i.e., the beginning of the material's magnetic polarity reversal along the outer perimeter occurs sooner that the saturation of the material along the inner perimeter. This case is optimal, since the function $B = f(F)$, for such a core geometry, is linear on a significant portion.

For this case, the following condition must hold

$$\alpha < \frac{h_s}{h_p} \qquad (11)$$

or

$$\alpha < \frac{F_s}{F_p}. \qquad (12)$$

To determine the function $B = f(F)$, we substitute Eqs. (9) and (10) in Eq. (8). For different values of F, the function $B = f(F)$ has segments with different characters of variation.

First segment. For the values $-F_{max} \leq F \leq F_p$,

$$B_1 = \frac{1}{R_0 - R_1} \int_{R_1}^{R_0} -b_s dR, \quad B_1 = -b_s. \quad (13)$$

Second segment. For the values $F_p \leq F \leq \alpha F_p$,

$$B_2 = \frac{1}{R_0 - R_1} \left[\int_{R_1}^{R} \frac{2b_s}{F_s - F_p} \left(F \frac{R_1}{R} - \frac{F_s + F_p}{2} \right) dR + \right.$$

$$\left. + \int_{R}^{R_0} -b_s dR \right]$$

(here, $R = R_1 F / F_p$ is the radius dividing the core into two parts, in one of which the process of magnetic polarity reversal has already begun, while in the other it has not).

$$B_2 = \frac{1}{\alpha - 1} \frac{b_s}{F_s - F_p} \left[2F \left(\ln \frac{F}{F_p} - 1 \right) - \right.$$

$$\left. - F_s (\alpha - 1) + F_p (\alpha + 1) \right]. \quad (14)$$

Third segment. For the values $\alpha F_p \leq F \leq F_s$,

$$B_3 = \frac{1}{R_0 - R_1} \int_{R_1}^{R_0} \frac{2b_s}{F_s - F_p} \left(F \frac{R_1}{R} - \frac{F_s + F_p}{2} \right) dR,$$

$$B_3 = \frac{1}{\alpha - 1} \frac{b_s}{F_s - F_p} \left[2F \ln \alpha - (F_s + F_p)(\alpha - 1) \right]. \quad (15)$$

Fourth segment. For the values $F_s \leq F \leq \alpha F_s$,

$$B_4 = \frac{1}{R_0 - R_1} \left[\int_{R_1}^{R} b_s dR + \right.$$

$$\left. + \int_{R}^{R_0} \frac{2b_s}{F_s - F_p} \left(F \frac{R_1}{R} - \frac{F_s + F_p}{2} \right) dR \right]$$

(here, $R = R_1 F / F_s$ is the radius dividing the core into two parts, in one of which saturation has already occurred, while it has not in the other),

$$B_4 = \frac{1}{\alpha - 1} \frac{b_s}{F_s - F_p} \left[2F \left(1 + \ln \frac{\alpha F_s}{F} \right) - \right.$$

$$\left. - F_s (\alpha + 1) - F_p (\alpha - 1) \right]. \quad (16)$$

Fifth segment. For the values $\alpha F_s \leq F \leq F_{max}$ and for the reverse variation of F to value zero,

$$B_5 = \frac{1}{R_0 - R_1} \int_{R_1}^{R_0} b_s dR, \quad B_5 = b_s. \quad (17)$$

For negative ampere-turns of magnetization, the function $B = f(F)$ is determined analogously.

By increasing the quantity $\alpha (\alpha = R_0 / R_1)$, we obtain a second case, when $\alpha F_p = F_s$. This case is characterized by the absence of a linear portion of the function $B = f(F)$ on the ascending and descending arms of the magnetic polarity reversal loop.

A further increase in α leads to the third case, when $\alpha F_p > F_s$. With this, the function $B = f(F)$ has linear portions when the core has its magnetic polarity reversed in the narrow ring extending from the inner layers to the outer ones. In these last two cases, the limiting magnetic polarity reversal loops are determined by the same method as in the first case. For purposes of comparison, Fig. 8 shows the limiting magnetic polarity reversal loops for the first, second, and third cases of values of α:

$$\alpha = 0.75 \frac{h_s}{h_p}, \quad \alpha = \frac{h_s}{h_p}, \quad \alpha = 1.5 \frac{h_s}{h_p}$$

(curves I, II, III, respectively).

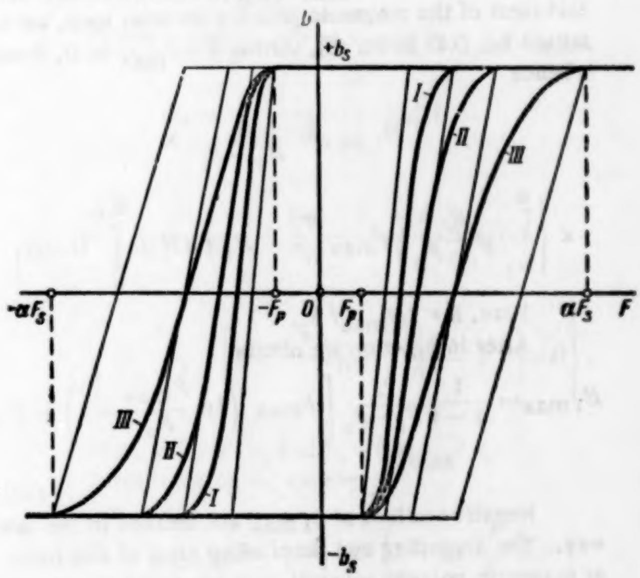


Fig. 8.

b. Magnetic Polarity Reversal by a Symmetric Loop ($F_{max} < \alpha F_s$)

When the maximum value of the magnetizing ampere-turns is inadequate for the magnetic polarity reversal of all the layers of the core material by the limiting hysteresis loop, the core will have its magnetic polarity reversed by symmetric loops. In connection with the circumstance that the individual layers of core material will have their magnetic polarities reversed by different symmetric cycles of the hysteresis loop, the mean maximum variation of the induction of the entire core B_{max} will be less than b_s. For the determination of the symmetric magnetic polarity reversal loops, it is necessary to determine, in addition to the ascending and descending arms of the magnetic polarity reversal loop,

the value of $\pm B_{max}$. As a function of the value of F_{max}, the symmetric magnetic polarity reversal loops will differ, not only in magnitude, but also by configuration, where this configuration will also depend on the core's geometry.

We consider only the first case, where $\alpha F_p < F_s$. With further increase in α, the picture does not change qualitatively. To determine the function $B = f(F)$ for symmetric magnetic polarity reversal loops, it is necessary that Eqs. (5) and (6) be transformed to:

$$b = \frac{b_s}{F_s - F_p}\left[(2F - F_{max})\frac{R_1}{R} - F_p\right], \tag{18}$$

$$b = \frac{b_s}{F_s - F_p}\left[(2F + F_{max})\frac{R_1}{R} + F_p\right]. \tag{19}$$

As a function of the size of F_{max}, the symmetric magnetic polarity reversal loops will have three forms.

First Form of the Symmetric Magnetic Polarity Reversal Loops ($F_p < F_{max} \leq \alpha F_p$)

To determine B_{max} of the function $B = f(F)$ for this form of the magnetic polarity reversal loop, we substitute Eq. (18) in Eq. (8), setting $F = F_{max}$ in it, from whence

$$B_{1\,max} = \frac{1}{R_0 - R_1} \times$$

$$\times\left\{\int_{R_1}^{R} \frac{b_s}{F_s - F_p}\left(F_{max}\frac{R_1}{R} - F_p\right)dR + \int_{R}^{R_0} 0 \cdot dR\right\}.$$

Here, $R = R_1 F_{max}/F_p$.

After integration we obtain

$$B_{1\,max} = \frac{1}{\alpha - 1}\frac{b_s}{F_s - F_p}\left[F_{max}\left(\ln\frac{F_{max}}{F_p} - 1\right) + F_p\right]. \tag{20}$$

Negative values of $B_{1\,max}$ are defined in the same way. The ascending and descending arms of this form of magnetic polarity reversal loop are found by substituting Eqs. (18) and (19) in Eq. (8).

By knowing the ascending and descending arms, and also $\pm B_{1\,max}$, one can construct the entire magnetic polarity reversal loop.

First segment. For values of $-F_{max} \leq F \leq F_p$,

$$B_1 = -B_{1\,max}. \tag{21}$$

Second segment. For values of $F_p \leq F \leq F_{max}$,

$$B_2 = \frac{1}{R_0 - R_1}\left\{\int_{R_1}^{R} \frac{b_s}{F_s - F_p} \times\right.$$

$$\times\left[(2F - F_{max})\frac{R_1}{R} - F_p\right]dR +$$

$$\left.+\int_{R}^{R_1} -\frac{b_s}{F_s - F_p}\left(F_{max}\frac{R_1}{R} - F_p\right)dR + \int_{R_1}^{R_0} 0 \cdot dR\right\}.$$

Here,

$$R = R_1\frac{F}{F_p},\quad R_1 = R_1\frac{F_{max}}{F_p}.$$

After integration, we get

$$B_2 = \frac{1}{\alpha - 1}\frac{b_s}{F_s - F_p}\left[2F\left(\ln\frac{F}{F_p} - 1\right) + \right.$$

$$\left.+ F_{max}\left(1 - \ln\frac{F_{max}}{F_p}\right) + F_p\right]. \tag{22}$$

Third segment. For values of $-F_p \leq F \leq F_{max}$,

$$B_3 = B_{1\,max}. \tag{23}$$

For negative values of F, the corresponding portions of the function $B = f(F)$ are determined analogously.

Second Form of the Symmetric Magnetic Polarity Reversal Loops ($\alpha F_p \leq F_{max} \leq F_s$)

We first determine B_{max} of the function $B = f(F)$ for this form of the magnetic polarity reversal loop:

$$B_{2\,max} = \frac{1}{R_0 - R_1}\int_{R_1}^{R_0} \frac{b_s}{F_s - F_p}\left(F_{max}\frac{R_1}{R} - F_p\right)dR$$

or

$$B_{2\,max} = \frac{b_s}{F_s - F_p}\left(\frac{\ln\alpha}{\alpha - 1}F_{max} - F_p\right). \tag{24}$$

We now determine the ascending and descending arms of the function $B = f(F)$ for this form of the magnetic polarity reversal loop.

First segment. For values of $-F_{max} \leq F \leq F_p$,

$$B_1 = -B_{2\,max}$$

Second segment. For values of $F_p \leq F \leq \alpha F_p$,

$$B_2 = \frac{1}{R_0 - R_1}\left\{\int_{R_1}^{R} \frac{b_s}{F_s - F_p} \times\right.$$

$$\times\left[(2F - F_{max})\frac{R_1}{R} - F_p\right]dR -$$

$$\left.-\int_{R}^{R_0} \frac{b_s}{F_s - F_p}\left(F_{max}\frac{R_1}{R} - F_p\right)dR\right\}.$$

Here, $R = R_1 F/F_p$.

After integration, we get

$$B_2 = \frac{1}{\alpha - 1}\frac{b_s}{F_s - F_p}\left[2F\left(\ln\frac{F}{F_p} - 1\right) - \right.$$

$$\left.- F_{max}\ln\alpha + F_p(\alpha + 1)\right]. \tag{25}$$

Third segment. For values of $\alpha F_p \leq F \leq F_{max}$,

$$B_3 = \frac{1}{R_o - R_i} \times$$

$$\times \int_{R_i}^{R_o} \frac{b_s}{F_s - F_p}\left[(2F - F_{max})\frac{R_i}{R} - F_p\right] dR,$$

$$B_3 =$$

$$= \frac{1}{\alpha-1}\frac{b_s}{F_s-F_p}[(2F - F_{max})\ln\alpha - F_p(\alpha-1)]. \qquad (26)$$

Fourth segment. For values of $-F_p \le F \le F_{max}$,

$B_4 = B_2$ max.

For negative values of F, the descending arm is determined analogously.

Third Form of the Symmetric Magnetic Polarity Reversal Loops ($F_s \le F_{max} \le \alpha F_s$)

We determined B_{max} of the function $B = f(F)$ for this form of the magnetic polarity reversal loop:

$$B_{3\max} = \frac{1}{R_o - R_i}\left[\int_{R_i}^{R_1} b_s dR + \int_{R_1}^{R_o} \frac{b_s}{F_s-F_p}\left(F_{max}\frac{R_i}{R} - F_p\right)dR\right].$$

Here, $R_1 = R_i F_{max}/F_s$.
Integration gives

$$B_{3\max} = \frac{1}{\alpha-1}\frac{b_s}{F_s-F_p}\times$$

$$\times\left[F_{max}\left(1+\ln\frac{\alpha F_s}{F_{max}}\right) - F_s - F_p(\alpha-1)\right]. \qquad (27)$$

We now determine the ascending and descending arms of the function $B = f(F)$ for this form of magnetic polarity reversal loop.

First segment. For the values $-F_{max} \le F \le F_p$,

$B_1 = -B_3$ max.

Second segment. For values $F_p \le F \le \alpha F_p$,

$$B_2 = \frac{1}{R_o-R_i}\left\{\int_{R_i}^{R_1}\frac{2b_s}{F_s-F_p}\left(F\frac{R_i}{R} - \frac{F_s+F_p}{2}\right)dR + \right.$$

$$+\int_{R_1}^{R}\frac{b_s}{F_s-F_p}\left[(2F-F_{max})\frac{R_i}{R}-F_p\right]dR +$$

$$\left. + \int_R^{R_o} -\frac{b_s}{F_s-F_p}\left(F_{max}\frac{R_i}{R}-F_p\right)dR\right\}.$$

Here, $R_1 = R_i F_{max}/F_p$ and $R = R_i F/F_p$. After integration, we obtain

$$B_2 = \frac{1}{\alpha-1}\frac{b_s}{F_s-F_p}\left[2F\left(\ln\frac{F}{F_p}-1\right) - \right.$$

$$\left. - F_{max}\left(1+\ln\frac{\alpha F_s}{F_{max}}\right) + F_s + F_p(\alpha+1)\right]. \qquad (28)$$

Third segment. For values of $\alpha F_p \le F \le F_s$,

$$B_3 = \frac{1}{R_o-R_i}\left\{\int_{R_i}^{R_1}\frac{2b_s}{F_s-F_p}\left(F\frac{R_i}{R}-\frac{F_s+F_p}{2}\right)dR + \right.$$

$$\left. + \int_{R_1}^{R_o}\frac{b_s}{F_s-F_p}\left[(2F-F_{max})\frac{R_i}{R}-F_p\right]dR\right\}.$$

Here, $R_1 = R_i F_{max}/F_s$.
Integration gives

$$B_3 = \frac{1}{\alpha-1}\frac{b_s}{F_s-F_p}\left[2F\ln\alpha - \right.$$

$$\left. - F_{max}\left(1+\ln\frac{\alpha F_s}{F_{max}}\right)+F_s - F_p(\alpha-1)\right]. \qquad (29)$$

Fourth segment. For values of $F_s \le F \le F_{max}$,

$$B_4 = \frac{1}{R_o-R_i}\left\{\int_{R_i}^{R} b_s dR + \right.$$

$$+ \int_R^{R_1}\frac{2b_s}{F_s-F_p}\left(F\frac{R_i}{R}-\frac{F_s+F_p}{2}\right)dR +$$

$$\left. + \int_{R_1}^{R_o}\frac{b_s}{F_s-F_p}\left[(2F-F_{max})\frac{R_i}{R}-F_p\right]dR\right\}.$$

Here, $R = R_i F/F_s$ and $R_1 = R_i F_{max}/F_s$.
After integration, we obtain

$$B_4 = \frac{1}{\alpha-1}\frac{b_s}{F_s-F_p}\left[2F\left(1+\ln\frac{\alpha F_s}{F}\right) - \right.$$

$$\left. - F_{max}\left(1+\ln\frac{\alpha F_s}{F_{max}}\right) - F_s - F_p(\alpha-1)\right]. \qquad (30)$$

Fifth segment. For values of $F_{max} \le F$ and $-F_p \le F \le F_{max}$,

$B_5 = B_3$ max.

For negative values of F, the descending arm is found analogously.

By giving various values of F_{max} we can, by Eqs. (20), (24), and (27), compute the "basic magnetization curve." Figure 9 gives the computed family of symmetric magnetic polarity reversal loops and the "basic magnetization curve."

c) Magnetic Polarity Reversal by a Biased Loop ($F_= \ge \alpha F_s$)

We now consider the mode of magnetic polarity reversal where, at the initial position, all of the core material is magnetized by saturation $F_= \geq \alpha F_s$ (Fig. 6 for $h_= \geq h_s$). In this mode, the magnetization proceeds by the ascending arm of the limiting magnetic polarity reversal loop and returns to the original state by the descending arm of a biased loop. We shall consider the character of the variation of the function B = f(F) only for the reverse magnetic polarity reversal (the return). For the determination of the function B = f(F), it is necessary to transform Eq. (7) to the form

$$b = -b_s - \frac{2b_s}{F_s - F_p}\left(F' - F_{\max}\right)\frac{R_1}{R}. \tag{31}$$

Here, $F' = F - F_=$ are the ampere-turns for magnetization (after compensation of the constant magnetizing ampere-turns, the working point moves from A to A_1): $F'_{\max} = F_{\max} - F_=$.

As in the case of symmetric magnetic polarity re- versal loops, the biased loops, as a function of the valu of $F'_{\max}$, will have various magnitudes and configuratio We shall therefore distinguish the individual forms of th biased magnetic polarity reversal loops.

First Form of the Biased Magnetic Polarit Reversal Loops ($F_p \leq F'_{\max} \leq \alpha F_p$)

We substitute Eq. (31) in Eq. (8). The function B = f(F) on the descending arm of a biased magnetic polarity reversal loop will have individual functional segments, determined by the magnitude of F'.

First segment. For values of $-F_p \geq F' \leq F'_{\max}$,

$$B_1 = B_{\max}.$$

Second segment. For values of $-F_p \leq F' \leq -F'_{\max}$

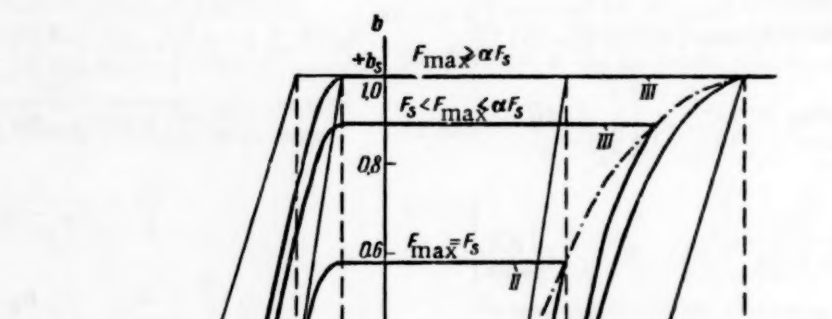


Fig. 9. Family of symmetric magnetic polarity reversal loops and the "basic magnetization curve" computed for the values $\alpha = 2$, $F_p = 0.2$ ampere-turns, $F_s = 0.8$ ampere-turns. I is the first form of the magnetic polarity reversal loop; II and III are the second and third forms, respectively.

$$B_2 = \frac{1}{R_o - R_i} \times$$

$$\times \left\{ \int_{R_1}^{R} \left[-b_s - \frac{2b_s}{F_s - F_p} (F' - F'_{max}) \frac{R_1}{R} \right] dR + \right.$$

$$+ \int_{R}^{R_1} \frac{2b_s}{F_s - F_p} \left(F'_{max} \frac{R_1}{R} - \frac{F_s + F_p}{2} \right) dR +$$

$$\left. + \int_{R_1}^{R_o} - b_s dR \right\}.$$

Here, $R = R_i F'/F_p$ and $R_1 = R_i F'_{max}/F_p$. Integration gives

$$B_2 = \frac{1}{\alpha - 1} \frac{b_s}{F_s - F_p} \left[2F' \left(1 - \ln \frac{F'}{F} \right) + \right.$$

$$+ 2F'_{max} \left(\ln \frac{F'_{max}}{F_p} - 1 \right) -$$

$$\left. - (F_s - F_p)(\alpha - 1) \right]. \quad (32)$$

Third segment. For values of $-F'_{max} \le F' \le -F_{max}$,

$$B_3 = -b_s.$$

Second Form of the Biased Magnetic Polarity Reversal Loops ($\alpha F_p \le F'_{max} \le F_s$)

First segment. For values of $-F_p \ge F' \le F'_{max}$,

$$B_1 = B_{max}.$$

Second segment. For values of $-F_p \le F' \le -\alpha F_p$,

$$B_2 = \frac{1}{R_o - R_i} \left\{ \int_{R_i}^{R} \times \right.$$

$$\times \left[-b_s - \frac{2b_s}{F_s - F_p} (F' - F'_{max}) \frac{R_1}{R} \right] dR +$$

$$\left. + \int_{R}^{R_o} \frac{2b_s}{F_s - F_p} \left(F'_{max} \frac{R_1}{R} - \frac{F_s + F_p}{2} \right) dR \right\}.$$

Here, $R = R_i F'/F_p$; $R_1 = R_i \frac{F'_{max}}{F_p}$.
After integration we get

$$B_2 = \frac{1}{\alpha - 1} \frac{b_s}{F_s - F_p} \left[2F' \left(1 - \ln \frac{F'}{F_p} \right) + \right.$$

$$\left. + 2F'_{max} \ln \alpha - F_s(\alpha - 1) - F_p(\alpha + 1) \right]. \quad (33)$$

Third segment. For values of $-\alpha F_p \le F' \le -F'_{max}$,

$$B_3 = \frac{1}{R_o - R_1} \times$$

$$\times \int_{R_1}^{R_o} \left[-b_s - \frac{2b_s}{F_s - F_p} (F' - F'_{max}) \frac{R_1}{R} \right] dR$$

or

$$B_3 = \frac{1}{\alpha - 1} \frac{b_s}{F_s - F_p} \times$$

$$\times \left[(2F'_{max} - 2F') \ln \alpha - (F_s - F_p)(\alpha - 1) \right]. \quad (34)$$

Fourth segment. For values of $-F'_{max} \le F' \le -F_{=}$,

$$B_4 = -b_s.$$

Third Form of the Biased Magnetic Polarity Reversal Loops ($F_s \le F'_{max} \le \alpha F_s$)

First segment. For values of $-F_p \le F' \le \alpha - F_p$,

$$B_1 = B_{max}.$$

Second segment. For values of $-F_p \le F' \le \alpha - F_p$,

$$B_2 = \frac{1}{R_o - R_1} \times$$

$$\times \left\{ \int_{R_1}^{R_1} - \frac{2b_s}{F_s - F_p} \left(F' \frac{R_1}{R} - \frac{F_s + F_p}{2} \right) dR + \right.$$

$$+ \int_{R_1}^{R} \left[-b_s - \frac{2b_s}{F_s - F_p} (F' - F'_{max}) \frac{R_1}{R} \right] dR +$$

$$\left. + \int_{R}^{R_o} \frac{2_s}{F_s - F_p} \left(F'_{max} \frac{R_1}{R} - \frac{F_s + F_p}{2} \right) dR \right\}.$$

Here, $R = R_i F'/F_p$ and $R_1 = R_i F'_{max}/F_s$. Integration gives

$$B_2 = \frac{1}{\alpha - 1} \frac{b_s}{F_s - F_p} \left[2F' \left(1 - \ln \frac{F'}{F_p} \right) + \right.$$

$$+ 2F'_{max} \left(1 + \ln \frac{\alpha F_s}{F'_{max}} \right) -$$

$$\left. - (F_s - F_p)(\alpha + 1) \right]. \quad (35)$$

Third segment. For values of $-\alpha F_p \le F' \le -F_s$,

$$B_3 = \frac{1}{R_o - R_1} \times$$

$$\times \left\{ \int_{R_1}^{R_1} - \frac{2b_s}{F_s - F_p} \left(F' \frac{R_1}{R} - \frac{F_s + F_p}{2} \right) dR + \right.$$

$$\left. + \int_{R_1}^{R_o} \left[-b_s - \frac{2b_s}{F_s - F_p} (F' - F'_{max}) \frac{R_1}{R} \right] dR \right\}.$$

Here, $R_1 = R_i F'_{max}/F_s$.
After integration we obtain

$$B_3 = \frac{1}{\alpha - 1} \frac{b_s}{F_s - F_p} \left[-2F' \ln \alpha + \right.$$

$$\left. + 2F'_{max} \left(1 + \ln \frac{\alpha F_s}{F'_{max}} \right) - F_s(\alpha + 1) + F_p(\alpha - 1) \right]. \quad (36)$$

Fourth segment. For values of $-F_s \le F' \le -F'_{max}$,

$$B_4 = \frac{1}{R_0 - R_1}\left\{\int_{R_1}^{R} - b_s dR - \right.$$

$$- \int_{R}^{R_1} \frac{2b_s}{F_s - F_p}\left(F'\frac{R_1}{R} - \frac{F_s + F_p}{2}\right) dR +$$

$$\left. + \int_{R_1}^{R_0}\left[-b_s - \frac{2b_s}{F_s - F_p}\left(F' - F'_{max}\right)\frac{R_1}{R}\right] dR\right\}.$$

Here, $R = R_1 F'/F_s$ and $R_1 = R_1 F'_{max}/F_s$.

Integration gives

$$B_4 = \frac{1}{\alpha - 1}\frac{b_s}{F_s - F_p}\left[-2F'\left(1 + \ln\frac{\alpha F_s}{F'}\right) + \right.$$

$$\left. + 2F'_{max}\left(1 + \ln\frac{\alpha F_s}{F'_{max}}\right) - (F_s - F_p)(\alpha - 1)\right]. \tag{37}$$

Fifth segment. For values of $-F'_{max} \le F' \le -F_=$,

$$B_5 = -b_s.$$

Figure 10 shows the family of biased magnetic polarity reversal loops.

By giving different values to $F_= \ge \alpha F_s$, we can compute the family of static characteristics $B = f(F, F_=)$ (Fig. 11).

Taking into account the nonuniformity of core magnetization as it affects its static characteristics when the material has a hysteresis loop of another form (for example, Figs. 1 and 3) can be done in an analogous fashion. Figure 12 shows the limiting magnetic polarity reversal loop of a core whose material has the hysteresis loop shown in Fig. 3.

CONCLUSIONS

On the basis of what has been presented here, we may draw the following conclusions.

1. When the material's hysteresis loop has the form shown on Fig. 2 or on Fig. 3, the core's magnetic characteristic, B = f(F), can be obtained in analytic form.

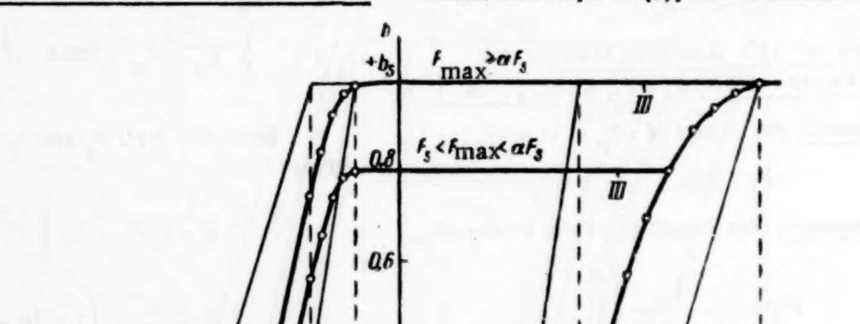


Fig. 10. Family of biased magnetic polarity reversal loops computed for the values $\alpha = 2$, $F_p = 0.2$ ampere-turns, $F_s = 0.8$ ampere-turns, $F_= = \alpha F_s =$ = 1.6 ampere-turns.

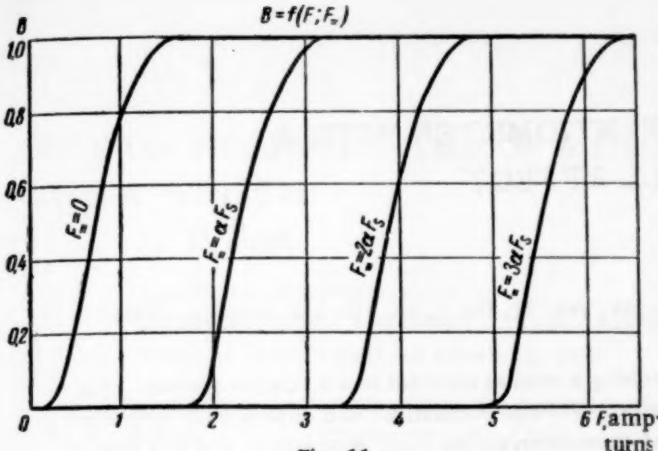


Fig. 11.

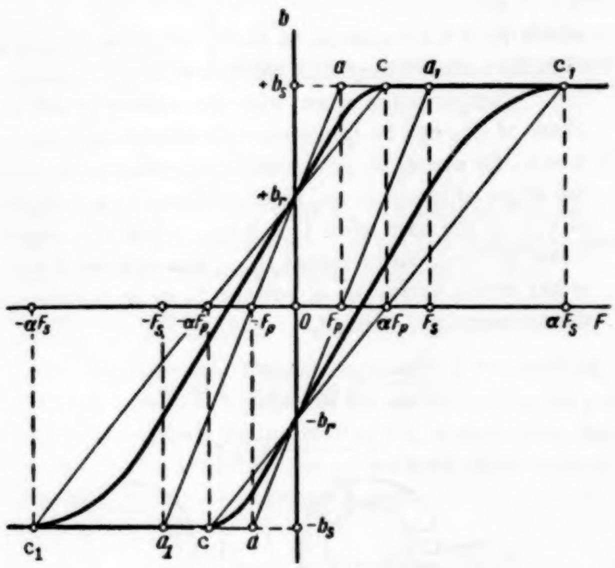


Fig. 12.

2. The form of a core's magnetic characteristic depends on its geometric relationships. There exist three cases of geometric relationships for cores which entail discrepancies in the forms of the magnetic characteristics (Figs. 7 and 8). For $\alpha F_p < F_s$ (the first case), the magnetic polarity reversal loop is characterized by linear segments on the ascending and descending arms. In this case, the outer layers of the core begin to reverse their magnetic polarity until saturation of the inner layers. For $\alpha F_p = F_s$ (the second case), the magnetic polarity reversal loop is characterized by an absence of linear portions on the ascending and descending arms. In this case, at the moment when reversal of the outer layer's magnetic polarity begins, the inner layer is already saturated. For $\alpha F_p > F_s$ (the third case), there is a broadening of the core's magnetic polarity reversal loop. In this case, the core's inner layers have become saturated even before reversal of the outer layers' magnetic polarity has begun. Linear portions of the ascending and descending arms of the magnetic polarity reversal loops are obtained due to the reversal of magnetic polarity of the material in a narrow toroidal sector, beginning at the moment when the inner layers saturate and continuing until the beginning of the reversal of the outer layers' magnetic polarity.

3. The form of the core's magnetic characteristic is also determined by the magnitude of the maximum ampere-turns for magnetization. When the magnitude of the magnetization ampere-turns is inadequate for complete magnetic polarity reversal of the entire core, the core's magnetic polarity is reversed by partial loops. In the modes of core magnetic polarity reversal by symmetric and biased partial loops, there are, as a function of the maximum magnetizing ampere-turns, three forms of loops, differing by their configurations and their dimensions (Figs. 9 and 10). The first loop form, when $F_p < F_{max} \le \alpha F_p$, is characterized by the fact that $\Delta B \ll 2b_s$, and also by the nonlinearity of the function $B = f(F)$ on the ascending and descending arms of the magnetic polarity reversal loop. In this case, only the inner layers have their magnetic polarity reversed. The second loop form, when $\alpha F_p < F_{max} \le F_s$, is characterized by the presence of a linear segment on the ascending and the descending arms of the magnetic polarity reversal loop, and by a larger ratio of $\Delta B / 2b_s$ than in the first case. In this case, the magnitude of the magnetizing ampere-turns is sufficient to reverse the magnetic polarity of the entire core, but inadequate for the saturation even of the inner layers. The third loop form, when $F_s < F_{max} \le \alpha F_s$, is characterized by the fact that all the core's layers have their magnetic polarity reversed, but not all are saturated. With this, the partial loops approximate to the form of the limiting core magnetic polarity reversal loop. All three magnetic polarity reversal loop forms have the same law of variation of the nonlinear segments on the ascending and descending arms.

4. The analytical methods presented can be conveniently employed also for other forms of material hysteresis loops. As an example, Fig. 12 shows the construction of a core's limiting magnetic polarity reversal loop for the material hysteresis loop shown on Fig. 3.

The taking into account of the effect of nonuniformity of core magnetization on its static characteristics for more complicated forms of the material's hysteresis loop, and also the experimental part of the work using ramified cores, will be considered in the second part of the present work.

LITERATURE CITED

[1] R. W. Roberts and R. J. Van Nice, "Influence of ID/OD ratio on static and dynamic magnetic properties of toroidal cores, Trans. AIEE, vol. 74, pt. I (1955).

[2] M. A. Rozenblat, "The dependence of the static characteristics of toroidal cores on their geometric dimensions," Avtomat. i Telemekh. 19, 8 (1958). †

† See English translation.

CONTACTLESS DC AUTOMATIC POTENTIOMETER WITH A TRANSFORMER BASED ON THE HALL EFFECT

V. N. Bogomolov

(Leningrad)

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The circuit is presented for a dc potentiometer lacking a normal element and a contact system. The potentiometer indications are virtually independent of supply voltage oscillations and depend only very slightly on the ambient temperature. Two identical Hall emf transducers are used as a transformer and as a source of compensating emf.

Automatic dc potentiometers are widely used today. In such potentiometers, the dc voltage $U_{in=}$ to be measured is compensated by a battery voltage, controlled by a normal element, until the difference becomes sufficiently small. Ordinarily, this difference is transformed to ac by a vibrapack, amplified, and then applied to a motor which moves the compensator's slide wire. The disadvantages of such circuits are the presence of the contact transformer, the slide wire with sliding contacts and the normal element. If a transformer based on the Hall effect [1] is used, then one contact element will be replaced by a contactless one. However, nothing is thereby changed in the remainder of the circuit, since the contact slide wire and the normal element remain. By using a Hall-effect transformer, it is nonetheless possible to construct a completely contactless potentiometer, in which there is no longer a necessity for a normal element, if one uses, for the compensation of the Hall emf, an identical transducer which is placed in a constant magnetic field and supplied by the line voltage.

The superiority of such a potentiometer over the ordinary ones consists in its greater mechanical stability, its greater reliability, simplicity, and length of serviceable life.

The circuit of this potentiometer is shown on the accompanying figure. The input dc signal $U_{in=}$ is immediately transformed to an ac signal $U_{x\sim}$ by the transformer based on the Hall effect, with transmission factor K_1. With this,

$$U_{x\sim} = K_1 U_{\text{in}\,=}.$$

If one takes into account that K_1 depends both on the temperature, t°, and on the supply (line) voltage U_s, then one may write

$$U_{x\sim} = K_1(t^\circ, U_s)\, U_{\text{in}\,=} = A f_1(t^\circ)\, f(U_s)\, U_{\text{in}\,=}.$$

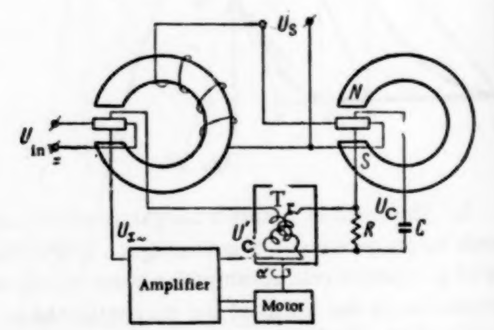
To compensate this alternating emf, one uses the voltage U_C, obtained from an identical transducer, but one immersed in a constant magnetic field and supplied from the line. The RC network is used to shift the phase of U_C by 90°. With this, one may also write

$$U_C = K_2(t^\circ)\, U_s = B f_2(t^\circ)\, f(U_s)\, H,$$

where K_2 is the transmission factor of this transducer and H is the constant magnetic field in which it is placed.

Compensation at ac, with the correct choice of phase of U'_C, can be implemented without contacts by means, for example, of a rotating transformer Tr, in which the angle of rotation, α, is proportional to the magnitude of U'_C. In the case when $U_{x\sim} = U'_C$, angle α is proportional to the input dc signal $U_{in=}$, and will not depend either on the variations of voltage U_s or on variations of the temperature. Indeed,

$$\frac{U_{x\sim}}{U_{C\sim}} = \frac{A f_1(t^\circ)}{B H f_2(t^\circ)}\, U_{\text{in}\,=}.$$



In the case of equality of the temperature dependencies, $f_1(t^\circ)$ and $f_2(t^\circ)$, the ratio just given depends only on $U_{in=}$ and, consequently, the transformer's angle of rotation α also depends only on $U_{in=}$. For transducers cut from one rod of germanium, $f_1(t^\circ)$ can differ from $f_2(t^\circ)$ by one to two percent which, in the temperature range of 70° and for a dependency $f(t^\circ)$ equal to 0.4%, gives an error at 1°C [2] for any angle α of rotation of 0.3 to 0.6%. This means that one can, with such a potentiometer, register a measured signal with a temperature error of 0.6% independently of the magnitude of the signal itself.

LITERATURE CITED

[1] V. N. Bogomolov, N. S. Nikolaenko, and V. P. Fedotov, "A transformer of dc to ac based on the Hall effect," Pribory i Tekh. Éksp. No. 2 (1959).*

[2] Collection: Germanium [Russian translation] ed. by D. A. Petrov (IL, 1955).

* See English translation.

NOTE ON THE PROPERTIES OF A SIMPLE EXTRAPOLATOR WITH A "SWITCH"

L. Prouza

(Pardubitse, Czechoslovakia)

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A theorem is proven which relates the properties of a simple extrapolator with a "switch" with lag present with the properties of an extrapolator without lag.

In [1] there was considered the stability of a simple extrapolator with a "switch." The transfer function of the continuous linear portion of such an extrapolator has the form:

$$Y(q) = e^{-\tau q}\left(\frac{\alpha}{q} + \frac{\beta}{q^2}\right) \tag{1}$$

and depends on the lag time τ. We shall now prove a theorem, of which theorem 1 of [1] is a particular case, and shall then consider several applications of it.

General Theorem

Let there be an extrapolator with a "switch" for which the transfer function of the continuous linear portion, with suitable normalization [1], is defined by formula (1). We introduce the generalized impulse transfer function of the open-loop system

$$Y^*(e^q, \varepsilon) = \sum_{n=0}^{\infty} y(n+\varepsilon)\, e^{-nq}, \tag{2}$$

where $y(t) = L^{-1}Y(q)$, $y(0) = 0$, $0 \leq \epsilon \leq 1$ [2, 3].

Let τ vary in the interval (0,1). One easily convinces oneself that, for $\epsilon \leq \tau$

$$Y^*(e^q, \varepsilon) = Y_1^*(e^q, \varepsilon) =$$

$$= \frac{[(\alpha - \beta\tau) + \beta(1+\varepsilon)]\, e^q - (\alpha - \beta\tau) - \beta\varepsilon}{(e^q - 1)^2} \tag{3}$$

and, for $\epsilon > \tau$

$$Y^*(e^q, \varepsilon) = Y_2^*(e^q, \varepsilon) = Y_1^*(e^q\varepsilon) + (\alpha - \beta\tau) + \beta\varepsilon. \tag{4}$$

From (3) and (4) we obtain the ordinary impulse transfer function

$$Y^*(e^q) = Y_1^*(e^q, 0) = e^{-q}Y_2^*(e^q, 1) =$$

$$= \frac{[(\alpha - \beta\tau) + \beta]\, z - (\alpha - \beta\tau)}{(z-1)^2}. \tag{5}$$

Here, we have used the notation $e^q = z$.

Theorem 1 (auxiliary). Let two extrapolators be given, the first with parameters α_1, β_1, and τ_1 and the second with parameters

$$\alpha_2 = \alpha_1 - \beta_1\tau_1, \qquad \beta_2 = \beta_1, \qquad \tau_2 = 0. \tag{6}$$

Then $Y^*(z)$ is identical for the two of them.

The proof follows immediately from the form of expression (5).

Theorem 2. All properties which depend only on transfer function (5) are identical for extrapolators whose parameters are related by relationships (6).

Theorem 2 is an obvious corollary of theorem 1.

Applications

It is clear that theorem 1 of [1] on the deformations of the region of stability is only a particular case of theorem 2.

In [4], one defined the error of tracking a "circular maneuver" for an extrapolator with $\tau = 0$ as a quantity proportional to the expression

$$E = \left|\frac{1}{1 + Y^*(e^{j\lambda})}\right| \tag{7}$$

[cf. [4], formula (30)]. Whence, by theorem 2, it is clear that, for an extrapolator with $\tau > 0$, it suffices to write $\alpha - \beta\tau$ instead of α in formulas (31), (32), and (33) in [4] in order to obtain the expression for the error for $\tau > 0$.

Further, in [4] there was defined a coefficient of error damping for the extrapolation of a unit velocity jump for $\tau = 0$, starting from the formula

$$E^*(z) = \frac{X^*(z)}{1 + Y^*(z)} \tag{8}$$

(cf. formula (8) in [4]), where $X^*(z) = z/(z-1)^2$. It follows from (8) and theorem 2 that the curves of constant damping coefficient (cf. [4], Fig. 4) for $\tau = 0$ are deformed as τ varies in the interval (0, 1) in accordance with relationships (6).

Further, let the noise coefficient of the extrapolator be defined by the formula (cf., for example, formula (53) in [5])

$$\sigma^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{Y^*(e^{j\lambda})}{1 + Y^*(e^{j\lambda})}\right|^2 d\lambda. \tag{9}$$

It is clear that expression (9) is identical for parameters related by relationships (6). By computing (9) (for example, by the method described in [6] or by formula (56) in [5]), we obtain, in the general case

$$\sigma^2 = \frac{2(\alpha-\beta\tau)^2+\beta[(\alpha-\beta\tau)+2]}{(\alpha-\beta\tau)[4-2(\alpha-\beta\tau)-\beta]}. \tag{10}$$

If we denote $\alpha - \beta\tau = A$, then the minimum of σ^2 for given β and τ is attained for

$$A = \sqrt{\beta} - \frac{\beta}{2}. \tag{11}$$

Formula (11), for $\tau = 0$, is given in [7]. It is clear from (10) that σ^2 tends to infinity on two sides of the triangle of stability. On the third side, corresponding to $\beta = 0$, σ^2 is defined by the formula

$$\sigma^2 = \frac{A}{2-A}. \tag{12}$$

Formula (12) can be derived from formula (86) in [5] where the case of an extrapolator with one integrator is considered in detail.

CONCLUSION

By means of relationships (6), one can, as a function of τ, transform not only the region of stability (theorem 1 in [1]), but other characteristics of the extrapolator system with "switch" considered here as well.

LITERATURE CITED

[1] L. Prouza, "On the stability of one extrapolating device," Avtomat. i Telemekh. 20, 4 (1959).*

[2] V. P. Perov, "Synthesis of sampled-data circuits and systems with sampled-data feedback," Avtomat. i Telemekh. 18, 12 (1957).*

[3] Ya. Z. Tsypkin, Transient and Steady-State Processes in Sampled-Data Circuits [in Russian] (Gosénergoizdat, 1951).

[4] J. Sklansky, "Optimizing the dynamic parameters of a track-while-scan system," RCA Rev. No. 2 (June, 1957).

[5] Ya. Z. Tsypkin, "Designing discontinuous control systems for stationary random stimuli," Avtomat. i Telemekh. 14, 4 (1953).

[6] G. W. Johnson, "Statistical analysis of sampled-data systems," Convent. Rec. IRE Wescon, pt. 4 (1957).

[7] A. H. Benner and R. Drenick, "An adaptive servo system," Convent. Rec. IRE, pt. 4 (1955).

* See English translation.

ELECTRETS AND THE PROSPECTS OF THEIR USE IN AUTOMATION

(Survey)

V. A. Andryushchenko

(Leningrad)

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The paper considers the present state of the question of electrets, which are the electrical analogs of magnets, and provides certain practical directions of their use as elements of automatic devices.

The creation of cheaper, more economical, relatively simple in design, and convenient to use automatic systems requires the use of new materials which possess new physical properties. Such a material is the "electret," which is little known in practice.

An electret is the electrical analog of a magnet. This is a dielectric possessing "constant" electrification with opposite charges at the ends.

The first specimens of electrets studied were mixtures of wax and gum whose congealing had occurred in a strong constant electrical field. The disk-shaped electrets thus prepared possess negative charges on the side turned to the anode and a positive charge on the opposite side. If an electret is bifurcated along a neutral line, then two electrets are obtained. Removal of an upper layer of an electret by scraping or shearing does not disturb the electret's properties.

Until today, essentially organic materials have been used for obtaining electrets: wax, gum, hydrocarbons, solid acids, alcohol, etc.

Inorganic materials (glass, porcelain, ceramics, etc.) have been little studied from the point of view of obtaining electrets from them, and the information thus far obtained about them has been contradictory.

TABLE

Material	$MgTiO_3$	$ZnTiO_3$	$BaO \cdot 4TiO_2$	Bismuth titanate	$CaTiO_3$	$SrTiO_3$	Strontium bismuth titanate	$BaTiO_3$
ε	16	22	28	80	150	175	750	1200

Electrets of inorganic materials have a higher rigidity and thermal stability than electets prepared of organic materials, and can have very diverse dielectric permeabilities (cf. the appended table).

For the preparation of electrets from organic materials there is no particular sense in increasing the temperature of the dielectric very much above the melting point. However, it is very important that the mass be completely converted to the liquid state before being placed in the electric field.

For the preparation of electrets, one ordinarily uses, as the electrodes for the electrization process, tin foil which is easily removed from the surface of the congealed specimens. It turned out that, with a congealed mixture of wax and gum, sufficiently good electrets could be obtained even in the absence of an external polarizing field. In this case, the size of the charge on the poles was 60 to 70% of the charge appearing on electrets obtained from placement in a polarizing field.

The electrets' properties depend, to a certain extent, on the magnitude of the electric field strength.

Once the field strength has reached a magnitude of 10 to 12 kw/cm, further increase of it does not induce any increase in the charge. This is explained by the circumstance that the electrical field on the electret surfaces reaches the maximum size which can exist in air for these gradients, i.e., about 5 CGSE/cm^2. Further increase of the field strength leads to ionization of the air molecules, and discharge commences. There is a corresponding limiting value to the possible density of the charge on the electret's surfaces. For normal atmospheric pressure, temperature, and humidity, the charge density cannot exceed $3 \cdot 10^{-5}$ coul/m^2. With increasing pressure the charge density also increases and, in regions of low pressure, the charge density is proportional to the pressure.

Electrets prepared from organic materials are decomposed by fusion. The quantity of electricity obtained by measuring the current flowing through the external circuit during the time of fusion does not depend on the time elapsed between polarization and fusion. It is almost directly proportional to the strength of the polarizing electrical field. The experimentally obtained maximum current with fusion attained a magnitude of about 10^{-8} amp/cm^2. When all of the specimen of the

electret had passed into the liquid state, the current ceased.

For storage, electrets of organic material are sealed. In a humid atmosphere the electret's charge drops sharply, but is reestablished on drying.

In the P. N. Lebedev Institute of Physics of the Academy of Sciences, USSR, the following mode of polarization was used for obtained electrets from inorganic material: The specimen was placed in an electrical field of 20 kw/cm strength, maintained at room temperature for 30 minutes, and then the temperature was raised to 200°C during 2 hours, after which the specimen was kept at this temperature for another 2 hours, and then the temperature was lowered to 60 to 80°C. The surfaces of the specimens and the electrodes (brass disks were used for the latter) were carefully polished.

The dimensions and the configurations of the specimens can be diverse. The magnitude of the field strength and the seasoning temperature for polarization are limited by the material's stability with respect to thermal disruption. Electrets of inorganic material can be stored either with or without sealing. The charges of electrets of organic and inorganic materials are retained for periods of years. The basic disadvantage of electrets today is the instability of their charge.

We turn now to photoelectrets, which were discovered in 1937 by the Bulgarian academician G. Nadjanoff.

As specimens, sulfur flakes (monocrystalloid or polycrystalloid) were used, these being polished to mirror smoothness by silk moistened with kerosene. A constant voltage of 470 v/cm was applied to a specimen, and the specimen, after three minutes of remaining in darkness, was exposed to a light of 6000 lux during 12 minutes, after which it was sealed and stored in darkness. The measurement of a photoelectret's charge was carried out by means of its depolarization with repeated illumination. A photoelectret can retain its charge for several days.

Illumination of a photoelectret at low temperatures (−80 to −100°C) leads only to its partial discharge, and complete depolarization occurs only as the result of increasing the specimen's temperature to the temperature at which its polarization occurred. Heating of a photoelectret without simultaneous illumination does not give rise to a discharge of current large enough to be measured, although the total charge on the photoelectret is somewhat decreased. The state of the polarized specimen just described may be called thermophotoelectret.

We now consider instruments (based on electrets) which are elements of automatic systems, or which execute functions of an automatic device.

Figure 1 shows a telephone circuit. If one speaks into microphone A, the sound waves start membrane 1 oscillating. The distance between it and electret 2 will change. As the membrane approaches the electret, the charge induced on it will increase and, as the membrane moves further away, the charge will decrease. These variations of charge induce currents in the telephone circuit which lead to acoustical vibrations of membrane 5 and receiver B. On Fig. 1, 3 is the immovable microphone electrode and 4 is the receiver coil.

One can create ac and dc generators using electrets. A dc generator is shown on Fig. 2 a. During an oscillatory swing of the upper electrode, when the electrodes are close key K_1 will make, i.e., seal the electrodes and key K_2 will break. The charge induced on the electrode by the short circuit will, as the latter moves away, be applied to the plates of condenser C; with this, key K_2 closes and key K_1 opens. By repeating this operation many times, one can charge the condenser up to a potential whose value will be limited only by the breakdown voltage of dielectric e between the condenser plates. In practice, the condenser was charged up to 5 kv.

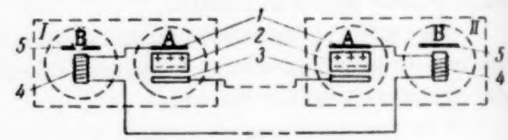
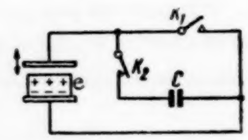


Fig. 1.



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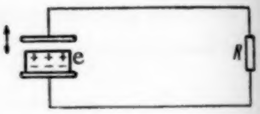

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Fig. 2.

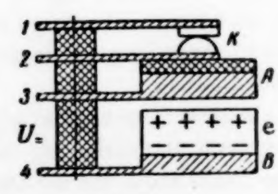

Fig. 3.

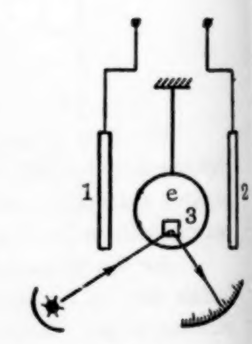

Fig. 4.

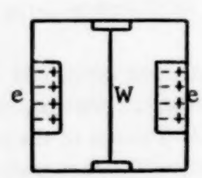

Fig. 5.

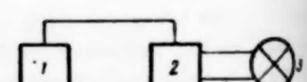

Fig. 6.

An ac generator (Fig. 2 b) is obtained by connecting resistor R in the electrical circuit. With oscillatory motion of the electrodes at a frequency of f = 100 cps, an ac current of 10^{-6} amp passes through load R; the effective power on the load in this case equals $P = 10^{-5}$ to 10^{-6} w.

An electrostatic relay based on the use of electrets, and shown schematically in Fig. 3, operates in the following way: As some constant voltage is applied to terminals 3 and 4, the upper electrode A is attracted to electret e attached to immovable electrode B. This causes contacts K to open. Thus, the relay will close or open (depending on the design) a second circuit connected to terminals 1 and 2.

If an electret is suspended on a thread between two metallic electrodes (Fig. 4) on which a charge has been placed, then the electrostatic fields of electrodes 1 and 2 and electret e will interact, turning the electret at a definite angle which is proportional to the charge on the electrodes. A small mirror 3 is fastened to the electret, by means of which mirror readings are made. The instrument, consequently, is a type of mirror galvanometer, in which an electret replaces a moving coil.

In a string voltmeter (Fig. 5) charged to the potential to be measured, wire W is placed between two electrets e which move against each other and have opposite charges. Since electrets change the magnitude of their charge in a radioactive field, after whose removal the charge is again reestablished, then one may judge the degree of radioactivity of a given field as a function of the electret's decrease in charge.

Photoelectrets can be used for danger signaling (Fig. 6). When photoelectret 1 is depolarized, current flows to device 2 which lights signal lamp 3.

Electrets can be used in electronic tubes for applying grid bias, and also in cathode-ray tubes for controlling the electron beam, and also in other instruments where only an electrostatic field is required.

Recently, there has been talk of using electrets in the memory devices of electronic computers.

The theoretical possibility of using electrets is based on the fact that the charges placed on the dielectric's surfaces do not spill over but, as it were, "stick." With this, if an electron falls on an electret's positive pole, it compensates the electret's charge in the given place.

If one could succeed in making an electret in the form of a fine long tape, on one side of which positive charges were placed, with negative charges on the other side, then one could "write," on such an electret tape, a series of definite pulses, using an electron beam. Such a tape, moved past another electron beam, could deflect it in correspondence with the variations in intensity of the first beam.

SUMMARY

The use of electrets in automatic devices allows:

1) the creation of new elements for automatic devices based on the physical properties of electrets;

2) simple design implementation of block schematics of automatic devices;

3) simplification of electric power suplies for automatic devices;

4) decreasing the weight and size of automatic devices.

LITERATURE CITED

[1] "Electrets," in: Problems of Modern Physics, ed. 17 [Russian translation] (IL, 1951).

[2] V. G. Nazarov, "Electrets and their use in electrical engineering," Élektrichestvo 7 (1954).

[3] A. N. Gubkin, "On the question of phenomenological theory of electrets," Zhur. Exptl. i Teoret. Fiz. 27, 9 (1957).

[4] A. N. Gubkin and G. I. Skanavi, "New electrets from inorganic dielectrics," Zhur. Exptl. i Teoret. Fiz. 32, 1 (1957).

[5] A. N. Gubkin and G. I. Skanavi, "Production and properties of new electrets from inorganic dielectrics," Izvest. Akad. Nauk SSSR, Ser. Fiz. 22, 3 (1958).

[6] I. S. Zheludev and V. M. Fridkin, "On photoelectronic and thermoelectronic states in multicrystalloid sulfur," Izvest. Akad. Nauk SSSR, Ser. Fiz. 22, 3 (1958).

[7] B. P. Sukhov, " Electrets," Nauka i Zhizn' 9 (1958).

[8] M. Eguchi, "Effects of electric forces in a permanent electret, Japan J. Phys. 1, 1 (1922).

[9] G. Nadjanoff, "Über eine Art von Elektreten: Photoelektreten,"Physik Z. Sowjetunion 36 (1938).

[10] A. Gremant, "The use of electrets in electrical instruments," Rev. Sci. Instr. 11 (1940).

[11] F. Gutmann, "The electret," Rev Modern Phys. 20 (1948).

[12] L. A. Freedman and L. A. Rosenthal, "Apparatus for the study of electrets," Rev. Sci. Inst. 21 (1950).

[13] Kato Kiyoe, "On the electrets," Natural Science Report of the Ocharomizu University, No. 2 (1955).

LETTERS TO THE EDITOR

A NOTE TO A PAPER OF YU. S. SOBOLEV, "ON THE ABSOLUTE STABILITY OF CERTAIN CONTROLLED SYSTEMS," [1]

(Translated from: Avtomatika i Telemekhanika, Vol. 21, No. 1, pp. 143-144, January, 1960))

In [1], the following theorem was actually proven (below, we retain the symbology and the formula numeration used in [1]).

If, for system (2), satisfying limitations "a"-"e", one can cite a positive definite function $r^2 = V(x_1, \ldots, x_n)$ not depending on h_k ($k = n-m+1, \ldots, n$), whose total derivative with respect to time is, by virtue of (2), negative definite for any $h_k \in (0, +\infty)$, then system (1), satisfying conditions "a"-"e", is absolutely stable.

In fact, in case r^2 depends on h_k, inequality (5) holds, in general, only for sufficiently small h_k ($k = n-m+1, \ldots, n$), and inequality (6) holds only for sufficiently large h_s and sufficiently small h_p. Therefore, there might not exist a system of values of h_k for which both inequalities (5) and (6) would be valid simultaneously. In exactly the same manner, it is impossible to assert anything as to the simultaneous holding of inequalities (7) or (9).

In the case of a linear system (2), the Routh-Hurwitz conditions do not guarantee the presence of a Lyapunov function, mentioned in the conditions of the theorem proven in [1]. We consider, for example, the equation

$$a_3 \frac{d^2x}{dt^2} + a_2 \frac{dx}{dt} + a_1 x = -\varphi(\sigma), \quad \sigma = b_2 \frac{dx}{dt} + b_1 x \tag{I}$$

($\varphi(\sigma)$ is a class A function).

Following [1], we replace (I) by the equation

$$a_3 \frac{d^2x}{dt^2} + a_2 \frac{dx}{dt} + a_1 x = -h\left(b_2 \frac{dx}{dt} + b_1 x\right), \; h \in (0, +\infty),$$

from which we go to the system

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -\frac{a_2 + hb_2}{a_3} x_2 - \frac{a_1 + hb_1}{a_3} x_1. \tag{II}$$

According to the Routh-Hurwitz conditions, the trivial solution of system (I) is asymptotically stable, at least for

$$a_1, a_2, a_3, b_1 > 0, \; b_2 \geqslant 0. \tag{III}$$

We now show that, among the systems (II) satisfying (III), there are such for which it is impossible to construct a positive definite function $V(x_1, x_2) = \sum_{ij=1}^{2} \alpha_{ij} x_i x_j$ with coefficients α_{ij} which do not depend on $\underline{h}$, whose complete derivative with respect to time, dV/dt, by virtue of this system would be negative definite for all $h \in (0, +\infty)$.

Indeed,

$$-0.5 \frac{dV}{dt} = \alpha_{12} \frac{a_1 + hb_1}{a_3} x_1^2 + \left(\alpha_{12} \frac{a_2 + hb_2}{a_3} + \alpha_{22} \frac{a_1 + hb_1}{a_3} - \alpha_{11}\right) x_1 x_2 + \left(\alpha_{22} \frac{a_2 + hb_2}{a_3} - \alpha_{12}\right) x_2^2$$

and the necessary and sufficient conditions that dV/dt be negative definite have the form:

$$\alpha_{22} \frac{a_2 + hb_2}{a_3} - \alpha_{12} > 0, \tag{IV}$$

$$\alpha_{12} \frac{a_1 + hb_1}{a_3} \left(\alpha_{22} \frac{a_2 + hb_2}{a_3} - \alpha_{12}\right) - \frac{1}{4}\left(\alpha_{12} \frac{a_2 + hb_2}{a_3} + \alpha_{22} \frac{a_1 + hb_1}{a_3} - \alpha_{11}\right)^2 > 0$$

$$h \in (0, +\infty). \tag{V}$$

For the holding of (IV) it is necessary ($h \to 0$) that $\alpha_{22} a_2 / a_3 - \alpha_{12} \geq 0$ and, for (V) ($h \to +\infty$), that $-(\alpha_{12} b_2 - \alpha_{22} b_1) = 0$, such that for systems (II)-(III), in which $b_2 a_1 - b_1 a_2 < 0$, (IV)-(V) may not hold for any $h \in (0, +\infty)$.

Thus, the theorem of Yu. S. Sobolev [1] is inadequate for solving the problem posed in [2] of comparing theories of absolute stability.

Yu. I. Alitov.

LITERATURE CITED

[1] Yu. S. Sobolev, "On the absolute stability of certain controlled systems," Avtomat. i Telemekh. 20, 4 (1959).*

[2] A. M. Letov, The Stability of Nonlinear Controlled Systems [in Russian] (Gostekhizdat, 1955).

* See English translation.

Dear Editor

We became acquainted with the paper of Yu. S. Sobolev, "On the absolute stability of certain controlled systems," published in number 4 of your journal for 1959. This paper gave rise to a number of remarks, which we provide forthwith.

In the first part of the paper [formulas (1)-(9)], Yu. S. Sobolev proves the following assertion: If there

exists a positive definite function $r^2(x_1, x_2, \ldots, x_n)$, not depending on the numbers h_k, whose derivative, by virtue of system (2) is negative definite for all $0 < h_k < \infty$, then the derivative of r^2, by virtue of system (1), is also negative definite. This assertion is obvious, and gives rise to no objections. Further (cf. the fine print, formulas (10)-(12)), the generally known properties of linear systems with constant coefficients are adduced, after which (cf. the normal print up to section 2) the incomprehensible assertion is made: " the existence of a function r^2 whose total derivative, by virtue of system (1), is negative definite, is guaranteed by the Routh-Hurwitz conditions." It is not clear what is meant by the Routh-Hurwitz conditions in the general case for systems (1) and (2).

In examples 1 and 2, Yu. S. Sobolev writes that, "according to what was proven above, the region of absolute stability (13) coincides with the region of asymptotic stability. . . of system . . . (14), where $h \in (0, +\infty)$." It is possible to deduce from this that Yu.S. Sobolev believes, in particular, that the following assertion has been proven: The region of absolute stability of a system described by the set of equations

$$\dot{x}_k = \sum_{\alpha=1}^{n}{}' b_{k\alpha} x_\alpha + n_k f(\sigma),$$

$$\sigma = \sum_{s=1}^{n} j_s x_s, \quad \sigma f(\sigma) > 0, \tag{A}$$

where $b_{k\alpha}$, j_s, and n_k are constants, coincides with the region of asymptotic stability of system (A) in which one sets $f(\sigma) = h\sigma$ for all $h > 0$.

As is well known, this assertion is untrue (cf., for example, the book of V.A. Pliss, Some Problems of Stability of Motion as a Whole). The author's error consists in the following. It is tacitly assumed that system (A), in which $f(\sigma) = h\sigma$, always has a Lyapunov function which does not depend on $\underline{h}$.

In connection with this, it is impossible to consider as proven this assertion in section 2 of Yu. S. Sobolev's paper.

E. N. Rozenvasser and V. A. Pliss

CHRONICLE

SEMINAR ON THE ENGINEERING APPLICATIONS OF MATHEMATICAL LOGIC (1958-1959)*

(Translated from: Avtomatika i Telemekhanika, Vol. 21, No. 1, pp. 145-148, January, 1960)

In the fall semester of 1958 and the spring semester of 1959, the seminar on the engineering applications of mathematical logic continued its work under the direction of Lecturer V. I. Shestakov. Taking part in the seminar were scientific workers, teachers, and engineers of a number of research institutes and universities, including the Institute of Automation and Remote Control, the Laboratory for Information Transmission Systems, the Department of Applied Mathematics of the Academy of Sciences, USSR, and the Moscow State University. There were 13 sessions of the seminar in all, in which 11 papers were presented and discussed; of these, 10 contained original results and one was a tutorial paper.

Part of the papers touched on general questions of mathematical logic, arising, to be sure, from the requirements of the engineering applications of mathematical logic but having, perhaps, an independent theoretical value. Such were the papers of A. D. Talantsev, G. N. Povarov, and the tutorial paper of V. P. Goncharov.

A. D. Talantsev read two papers. In the first of them, "On the use of several logical operators for the analysis and synthesis of schemes containing differentiating circuits," presented Oct. 2, 1958, A. D. Talantsev dealt with the questions of the analysis and synthesis of potential-pulse circuits in which differentiating circuits entered. He proposed an essentially new algebraic-logical method for investigating such circuits. The basis of the method is the definition of special operators, the application of which to the "and" and "or" functions allowed one to derive relationships forming interesting analogies with the formulas of mathematical analysis.

It was shown in the paper how the relationships derived could be used for the simplification of circuits containing differentiating circuits. The concept of "covariance" of logical functions was introduced, this concept being related to the substantive meaning of the operators introduced; this concept can sometimes be used for transforming circuits. The exposition of the new concepts was illustrated by the example of a logical circuit for recognizing the direction of motion of a quantized scale for a digital control system for a milling machine.

The second paper of A. D. Talantsev, "On the analysis and synthesis of certain electrical circuits by means of special logical operators", was given on Jan. 16, 1959. In this paper there was developed (with account being taken of remarks made by M. L. Tsetlin and V. I. Shestakov) a new method of investigating electrical circuits which was first presented by A. D. Talantsev on Oct. 2, 1958. Specifically, there was derived an expansion formula for $dF(x_1, x_2, \ldots, x_n)$, where F is an arbitrary Boolean function of n variables. The relationship obtained for dF lies beyond the boundaries of Boolean algebra. Further, there were defined the concepts of potential and of pulse logical variables, the concept of a homogeneous potential-pulse circuit, and the potential-pulse form describing such a circuit. The problem was posed of "integrating" potential-pulse forms, and a general method of "integrating" these forms was given. In conclusion, the example was considered of "integrating" the forms describing a circuit which recognizes the direction of motion of a quantized scale.

The work of A. D. Talantsev was published in the journal Avtomatika i Telemekhanika 20, 7 (1959).**

G. N. Povarov gave three papers at the seminar. On Oct. 31, 1958, in the paper "On group invariance of Boolean functions," G. N. Povarov considered the group $\mathfrak{T}_n$, transforming monotypic Boolean functions to other such functions. The concept was introduced of the invariance of Boolean functions with respect to subgroups of group $\mathfrak{T}_n$ as well as the concept of regions of invariance of Boolean functions with respect to subgroups of group $\mathfrak{T}_n$.

G. N. Povarov showed that the regions of invariance comprise a Boolean algebra, and form a structure dually homomorphic to the structure of the subgroups of group $\mathfrak{T}_n$; to the conjugate subgroups correspond monotypic regions of invariance. Results were given of an investigation of a type of region of invariance $D(\mathfrak{S}_n)$ with respect to the group $\mathfrak{S}_n \subset \mathfrak{T}_n$ of permutations of arguments of Boolean functions. Also investigated were the types of regions of invariance contained in $D(\mathfrak{S}_n)$. The study of regions of invariance is important for the theory of switching circuits, since it enables one better to understand the types of regularity characteristic of simple circuits. A brief exposition of the work was then published in Russian in the "Annalele Stuntifice ale Universitatu Al. I. Cuza" din Iasi" (Rumanian People's Republic) 4, 1, 39-44 (Sept. 1, 1958).

In the next paper, to which two seminar sessions were devoted (Feb. 27 and Mar. 6, 1959), "An abstract algebraic theory of cumulative nets,"*** G. N. Povarov presented a mathematical theory developed by him, of indirect interactions in control, communications, and pla

* The proceedings of the seminar prior to October, 1958, were presented Avtomat. i Telemekh. 18, 10 (1957) and 20, 1 (1959) [See English translation of 1959 issue].

** See English translation.

*** In the invitations to this session of the seminar, this paper of G. N. Povarov had the name, "On one mathematical method of investigating operations."

ning nets of a very wide class, including various switching circuits, radio communications nets and wire networks, transportation networks, target networks, etc.

The central concept of the theory is the general combinatorial concept of a cumulative net, which is why G. N. Povarov calls this the theory of cumulative nets. The idea of a cumulative net is analogous to the idea of a normalized graph, but more complicated. It characterizes a definite type of indirect interaction, and the interactions characteristic of the enumerated control, communications, and planning nets. The generality of the treatment is provided by the use of the theory of numeroids [the concept of numeroids was introduced by G. N. Povarov in a paper printed in the journal Uspekhi Matematicheskikh Nauk, 11, 5(71), 195-202, 1956].

A cumulative net on the numeroid N is defined as a system $\{M, \|a_{ij}\|, \{P_n\}\}$, consisting of a set $M = \{1, 2, \ldots, p\}, p^2$, a matrix $\|a_{ij}\|$ on N and the sequence of sets $P_n \subseteq \sum_{r=2}^{n+1} M^r$, where $n = 1, 2, \ldots, P_1 = M^2$, and $P_n \subseteq P_{n+1}$ for all n. The matrix $\|a_{ij}\|$ is called the primitive connectivity of the net, and the matrix $\|a_{ij}^{(n)}\|$, where

$$a_{kl}^{(n)} = \sum_{ki_1 i_2 \ldots i_{r-1} l \in P_n} a_{ki_1} a_{i_1 i_2} \ldots a_{i_{r-1} l},$$

is called the nth-degree connectivity of the net, and the sequence $\{\|a_{ij}^{(n)}\|\} = \|a_{ij}^{(1)}\|, \|a_{ij}^{(2)}\|, \ldots$ is the evolution of the net's connectivity. The analysis of a cumulative net consists in finding $\{\|a_{ij}^{(n)}\|\}$ from $\|a_{ij}\|$ for known M and $\{P_n\}$. The interpretation of these quantities is that $\|a_{ij}\|$ is the measure of the immediate interaction between elements of M, $\|a_{ij}^{(n)}\|$ is the measure of indirect interaction via not more than n-2 intermediate elements, and $\{\|a_{ij}^{(n)}\|\}$ is the successive development of indirect interaction. $\{P_n\}$ ("the development of the net") defines the method of combining immediate interactions into indirect ones.

G. N. Povarov also spoke of the matrix methods he had developed for the analysis of various cumulative nets; he investigated such properties of cumulative nets as equivalence, stability, the presence of feedback, etc. G. N. Povarov showed that the theory of cumulative nets allows one to reduce to one framework the results of many authors working on the questions of the analysis of control, communications, and planning nets and analogous problems, in particular, the results obtained in these domains by B. I. Aranovich, A. G. Lunts, O. Plekhl, A. Shimbel, Z. Prikhar, G. Frobenius, and others, and also previous results of G. N. Povarov himself relative to the theory of switching circuits and communications networks.

The third paper of G. N. Povarov, "Phenomenological and juridical aspects of logic in connection with the logical problems of engineering," given on May 15, 1959, contained opinions on the convenience and naturalness of the interpretation of logical algebra in switching theory and in other branches of engineering as an algebra of events, and not as an algebra of judgments (propositions).

On May 29, 1959, V. P. Goncharov presented the contents of a paper by Zemanek, "Solving the equations of switching algebra," printed in the journal Archiv der Elektrischen Übertragung 12, 1, 35-44 (Stuttgart, 1958). The paper considered methods of finding inverse Boolean functions and solving the equations of Boolean algebra. Zemanek denotes a Boolean function by y_Z^n, where z is the decimal ordinal number of the function, corresponding to the binary representation of the constituent units entering into the given function, and n is the number of Boolean variables.

For the finding of inverse Boolean functions, a new notation was introduced in the paper: P and !. P can assume either the value 0 or the value 1, i.e., it corresponds to an "indifferent" state, but $\bar{P}$ always takes the value opposite to that of P; rules of action are defined for P. ! means that it is impossible to obtain a solution. Although Zemanek states in the paper that no rules of action will be introduced for !, it follows from Zemanek's reasoning, as pointed out by V. I. Shestakov in the discussion of the paper, that he has implicitly introduced certain rules of action for the sign !. In considering various cases of finding inverse functions, Zemanek presents a schematic representation of two of them; in these schemes, the sign P represents an arbitrarily closed or open key. Further, Zemanek considers the equation

$$y_{Z1}^{n+1}(x_i, y) = y_{Z2}^{n+1}(x_i, y),$$

where $i = 1, 2, \ldots, n$ and $y = y_Z^n$ is the unknown Boolean function. This equation is transformed to the following:

$$yy_{I1}^n + \bar{y}y_{N1}^n = yy_{I2}^n + \bar{y}y_{N2}^n,$$

where

$$y_{I1}^n = y_{Z1}^{n+1}(x_i, 1), \quad y_{N1}^n = y_{Z1}^{n+1}(x_i, 0),$$

and y_{I_2} and y_{N_2} is the similar notation for

$$y_{Z2}^{n+1}(x_i, y).$$

From this, Zemanek obtains the solution for the initial equation:

$$y_Z^n = {!}\,(y_{I1} \not\equiv y_{I2})(y_{N1} \not\equiv y_{N2}) +$$
$$+ O\,(y_{I1} \not\equiv y_{I2})(y_{N1} \equiv y_{N2}) + 1\,(y_{I1} \equiv y_{I2})(y_{N1} \not\equiv y_{N2}) +$$
$$\overline{+ P\,(y_{I1} \equiv y_{I2})(y_{N1} \equiv y_{N2})},$$

where $\equiv$ is the equivalence sign and $\not\equiv$ is the sign of its negation.

This solution contains all possible values of the Boolean function being sought. Zemanek showed the ap-

plication of the method developed by him to the solution of equations in Boolean algebra.

The paper of Zemanek is, so far as is known, the first attempt to give an algebraic representation for the general solution of equations in Boolean algebra.

In the other portion of the papers there were treated various questions of the analysis and synthesis of switching circuits. Such were the papers of Yu.L. Sagalovich, B. Yu. Pil'chak and V. D. Kazakov.

On Oct. 17, 1958, Yu. L. Sagalovich gave the paper "On the number of types of symmetric contact (1,k)-poles." The author recalled that contact (1, k)-poles realize the basic sequence of functions of n variables, and he considers them as coinciding correctly to the numbering of the output poles (terminals). The number of different (1,k)-poles equals the number of combinations $m = C^k_\mu$, where $\mu = 2^{2n}-2$. On all m (1,k)-poles one can carry out the operation of substituting variables and (or) inverting (negating) some of them. The set of these operations forms a group G of order $2^n n!$. (1,k)-poles are considered to be of one type if they are obtained from each other by operations from the group G.

Using the methods of the theory of group representations, the author found the number, $N_{n,k}$, of types of (1,k)-poles of n variables:

$$N_{n,\,k} = \frac{1}{2^n n!} \Sigma\, n_c \chi_c,$$

where n_c is the number of elements of class C of group G, χ_c is the character of class C in the representation of group G by substitution matrices of order m. To determine χ_c, one seeks the cyclical structure of the substitutions of degree m induced in the group G by the elements of class C.

B. Yu. Pil'chak, during two seminar sessions (Nov. 21 and Nov. 28, 1958), spoke on, "The synthesis of quasi-nonrepeated contact circuits." She gave this name to circuits in which there are connected to each relay not more than one make, and not more than one break, contact. These circuits are synthesized by the method of superposition of subcircuits. The subcircuits are isolated by a method which is a generalization of the method of B. A. Trakhtenbrot which he suggested for finding subcircuits of nonrepeated circuits. B. Yu. Pil'chak introduced the concepts of K-conductance and K-realizability, and suggested the formula for K-conductance. These formulas and concepts are necessary for the synthesis of circuits by the superposition method, and are generalizations of the ordinary concepts of conductance, realizability, and conductance formulas.

On April 17, 1959, V. D. Kazakov gave his paper, "Finding the maximum number of simple implicants of an arbitrary symmetric logical function of n variables." The speaker first recalled that, in the majority of methods for the minimization of logical functions and, consequently, of switching circuits, it is necessary to seek minimal equivalents of the original function which consist of some set of simple implicants of the initial function. For the mechanization of this search for minimal equivalents, it is necessary to know the greatest number of simple implicants I_{max} which can appear with one arbitrarily chosen function of n logical variables; the memory of the minimizing device must be constructed on the basis of such a greatest number I_{max}.

After this, V. D. Kazakov presented the method he had developed for determining the greatest number I_{max} of simple implicants of an arbitrary symmetric function of n variables. He showed that, for such functions, I_{max} is found by the formulas

$$\max\,[C_n^\alpha C_\alpha^\beta] + 2C_n^{\frac{n-\alpha-1}{2}} \quad (\text{ for even } \; n-\alpha-1),$$

$$\max\,[C_n^\alpha C_\alpha^\beta] + C_n^{\frac{n-\alpha}{2}} + C_n^{\frac{n-\alpha-2}{2}} \quad (\text{ for odd } \; n-\alpha-1),$$

where n is the number of variables in the original function, and α and β are quantities for definite values of which the product $C^\alpha_n\, C^\beta_\alpha$ of numbers of combinations becomes maximal. The formulas just given are approximate ones ; V. D. Kazakov also obtained an exact formula, but it is very cumbersome. V. D. Kazakov determined the form of the functions having I_{max} simple implicants, and computed, by the method suggested, the value of I_{max} for n = 1, 2, . . . , 20.

The papers of V. R. Telesnin and B. Ya. Falevich were devoted to the descriptions of new contactless circuits, for whose synthesis mathematical logic was employed.

On December 19, 1958, V. R. Telesnin gave the paper, "The use of magnetic matrices for information processing," in which a device was presented which was destined for both the storage and the processing of information. The basis of this device is a plane magnetic matrix used for a memory device. The total number of active elements is no larger than in matrices designed only for information storage. For the processing of information, there were introduced two special rows to which information could be transmitted from the ordinary rows of the memory device in direct, or complemented, code. Such a device has the capability of realizing any logical functions of the arguments written in the places (bit-positions) of all the rows. The addition of four more special rows allowed the realization of addition, subtraction, and multiplication of the numbers written in the matrice's rows.

On Jan. 30, 1959, B. Ya. Falevich, in the paper, "Electronic machines for games of "wolves and sheep," spoke of the schematic implementation, begun under his direction, of the substantive algorithm he had found for this game.

The algorithm presented by the author is not the very simplest.

Many of the papers gave rise to lively discussion.

V. P. Goncharov

BIBLIOGRAPHY

A LIST OF LITERATURE ON MAGNETIC ELEMENTS OF AUTOMATION, REMOTE CONTROL, AND COMPUTING TECHNOLOGY FOR 1958*

(Translated from: Avtomatika i Telemekhanika Vol. 21, No. 1, pp. 149-158, January, 1960)

1. General Questions. Terminology. Bibliography

"Magnetic amplifier bibliography 1951-1956," Commun. and Electronics, No. 39, pp. 613-627 (November, 1958).

"Magnetic amplifier bibliography 1957-1958," Appl. Sub-Committee of the AIEE Magnetic Amplifier Committee.

"Proposed standards for core test methods for toroidal magnetic amplifier cores," Commun. and Electronics No. 38, pp. 524-535 (1958).

"Proposed standard terms and definitions for magnetic amplifiers," Commun. and Electronics No. 38, pp. 429-431 (1958), 3 figs., 1-title bibl.; disc. pp. 431-432, 5-title bibl.

H. F. Storm, "Recommended symbols for magnetic amplifier papers," Proc. C.M.A. (1958), Los Angeles; AIEE, pp. 62-66 (1958), 5 tables.

"Terms and definitions for magnetic amplifiers," Commun. and Electronics No. 38, pp. 429-437 (1958).

2. Ferromagnetic Materials. Cores

a) Magnetic Materials

A. Braginski, "Magnetic materials," [in Polish], Elektronika 5, 1-2, 3-41 (1958).

W. J. Carr, "Theory of ferromagnetic anisotropy," Phys. Rev. 108, pt. 2, 5, 1158-1163 (1957), 1 table.

C. W. Chen, "Temperature dependence of magnetic properties of silicon-iron," J. Appl. Phys. 29, 9, 1337-1343 (1958).

P. F. Davis, "An electrical analogue of magnetic domains," Brit. J. Appl. Phys. 28, 1, 19-20 (1957).

H. C. Fiedler et al., "Magnetic properties of cube textured silicon-iron magnetic sheet," J. Appl. Phys. 29, 3, 363-365 (1958).

J. F. Fritz and J. J. Clark, "Effects of temperature on magnetic properties of nickel-iron alloys," El. Manufacturing, 62, 5, 135-139 (1958).

D. I. Gordan et al., "Radiation effects in magnetic materials," Nucleonics 16, 73-77 (1958).

"Ferrite cores for magnetic memory matrices," [in German]. Archiv der Elektrischen Ubertragung 12, 6 (1958).

H. B. Harms, "An experimental study of magnetic materials for use in ultrahigh-temperature electronic transformers," Commun. and Electronics, No. 36, pp. 181-184 (1958), 8 figs., 2-title bibl.

C. Heck, "Magnetic materials and their use in electronics and communications engineering," [in German] ETZ-A 79, 23, 932-934 (1958), 31 figs.

C. Heck and H. Repiser, "Ferrites with rectangular hysteresis loops, and the testing of the memory properties," [in German] Radio Mentor No. 2, pp. 107-110 (1957).

C. P. Lascaro and A. L. Long, "Effect of high intensity radiation on electronic parts and materials," El. Manufacturing 62, 3, 119-121, 306 (1958), 5 tables, 4-title bibl.

D. H. Martin, "Magnetism in materials," Wireless World 64, pp. 28-30 (Jan. 1958); (February, 1958), pp. 70-74; (March, 1958), pp. 126-131; (April, 1958), pp. 178-180.

"Materials used in radio and electronic engineering. 5. Magnetic materials," J. Brit IRE 18, 449-464, (1958), 1 fig., 9 tables, 91-title bibl.

"A new power ferrite for flyback transformers by Allen-Bradley," Electronics 31, 28 (1958).

G. G. Palmer et al., "Magnetic properties and associated microstructure of zinc-bearing square-loop ferrites," J. Amer. Ceram. Soc. 40, 8, 256-262 (1957), 13 figs, 13-title bibl.

R. H. Pry and C. P. Bean, "Calculation of the energy loss in magnetic sheet materials using a domain model," J. Appl. Phys. 29, 3, 532-539 (1958).

H. Reinboth, The Technology and Applications of Magnetic Materials, [in German] (VEB Verlag Technik, Berlin, 1958) 379 pp.

V. H. Richter, "Ferromagnets with rectangular hysteresis loops, and their applications," [in German] Elektrotechnik Nos. 15/16, 93-96 (1958), 12 figs.

R. Skorski, "Annealing iron and steel (0.45%) in a magnetic field," [in Czech.], Prace Inst. Mech. 6, 21, 32 (1957).

Z. Tischer, "Magnetically soft materials on the basis of nonferrous metals," [in Czech.], Slaboproudy Obz. 19, 5, 329-337 (1958), 23 figs, 2 tables.

Z. Tischer, "Expression for the rectangularity of magnetically soft materials," [in Czech.], Slaboproudy Obz. 19, 10, 653-657 (1958), 18 figs., 6-title bibl.

A. Villachon, "Alloys with rectangular hysteresis loops," [in French], L'Onde El. 37, 360, 259-261 (1957).

J. L. Walter, "Magnetic properties of cube textured transformer sheet," J. Metals, pp. 509-511 (August, 1958).

G. W. Wiener and K. Detert, "Cube oriented magnetic sheet. A major advance in magnetic materials," J. Metals, pp. 507-508 (August, 1958).

H. P. Wijn and H. Heide, "Pulse-response properties of rectangular-loop ferrites," Proc. IRE, B-104, Suppl. 7, pp. 422-427 (1957), 10 figs., 7-title bibl.

b) Magnetization Processes and Dynamic Characteristics of Ferromagnetic Materials (Dissipation, Eddy Currents, Viscosity, etc.)

R. C. Barker, "Magnetization in tape wound cores," Proc. C.M.A. (1958), Los Angeles, AIEE, pp. 1-46 (1958), 34 figs., 1 table, 43-title bibl.

T. C. Chen and A. Papoulis, "Terminal properties of magnetic cores," Proc. IRE 46, pt. 1, 5, 839-849 (1958).

R. L. Conger, "Thin film magnetization reversal studies," Proc. C.M.A. (1958), Los Angeles, AIEE, pp. 444-453 (1958), 7 figs., 5-title bibl.

G. C. Feth, "Core-reset functions in magnetic-amplifier analysis. I. A core-reset function. II. Determination of amplifier performance," Commun. and Electronics No. 38, pp. 503-519 (1958).

K. Ganzhorn, "Computing rectangular hysteresis loops in ferrites," [in German], Z. angew. Phys. 10, 4 169-172 (1958).

A. V. Kienlin, "Toward an elucidation of the question of magnetic aftereffects in ferrites caused by electron diffusion," [in German], Z. angew Phys. 9, 5, 245-250 (1957).

E. W. Lee, "Eddy-current losses in thin ferromagnetic sheets," Proc. IEE 105 C, 8, 337-342 (1958), 4 figs., 24-title bibl.

R. E. Morgan, "Reactions of bistable B-H loops on magnetic amplifiers," Commun. and Electronics No.39, pp. 586-590 (1958), 14 figs., 7-title bibl.

T. D. Rossing and V. J. Korkowski, "Switching properties of permalloy cores of varying coercivity," Appl. Phys 29, 3, 479-480 (1958).

T. D. Rossing et al., "The switching characteristics of 4-79 permalloy cores with different anneals," IRE Trans. Electronic Computers, EC-7, 3, 228-231 (1958), 8 figs.

T. D. Rossing and S. M. Rubens, "Effect of a transverse field on switching rates of magnetic memory cores," J. Appl. Phys. 29, 8, 1245-1247 (1958), 4 figs., 4-title bibl.

J. E. Wolf, "Influence of excitation frequency and waveform on squareness ratio of toroidal tape cores," Proc. C. M. A. pp. 94-112 (1958), 39 figs.

c) Physical Design and Technology of Preparation of Cores

R. E. Ball, "The effects of core variations on production quantities of magnetic amplifiers," Proc. C.M.A. (1958), Los Angeles, AIEE pp. 87-93 (1958), 5 figs. 3-title bibl.

C. Cook, "Some aspects of the design of toroidal inductors," T.M. C. Tech. J. 8, 3, 124-139 (1958).

G. J. Duncan and M. M. Felger, "Properties of electrical insulation at ultrahigh temperatures," Applic. and Ind. 77, pt. 2, 39, 300-305 (1958).

"Here's how magnetic amplifier design will be affected by tape wound core standarization," Electronics, Engineering Edition 31, 5, 59 (1958).

"High speed versatile toroid winders for production or laboratory," Electronics, Engineering Edition 31, 11, 275 (1958).

"How to make a magnetic core that's really small!" Electronics, Engineering Edition 31, 3, 44 (1958).

J. W. Kallender, "Designing minimum weight magnetic cores," El. Manufacturing 61, 2, 118-123 (1958), 3 figs.

T. Konorinskii, "Designing wound cores for communications devices," [in Polish] Przeglad Telekomunikacyjny 1, 1-7 (1958), 23 figs., 14-title bibl.

"Magnetic Metals Company processes tape wound centricores to provide thermal stability," Electronic Industries No. 3, p. 163 (1958).

D. W. Moore, "The preparation of magnetic films by high vacuum evaporation," Proc. C.M.A. (1958), Los Angeles, AIEE, pp. 278-295 (1958), 12 figs.

T. R. Nisbet, "Optimum design of power transformers and saturable reactors. Part I." Electronic Design, p. 4 (April, 1958).

"Now wind subminiature toroids automatically," Electronics, Engineering Edition 31, 11, 190 (1958).

F. T. Parr, "Organosilicon compounds as insulation for miniature power transformers for high temperature operation," Proc. Electronic Components Symposium, pp. 246-253 (1957).

A. C. Sim, "Nomogram for air-gap design," Electronic and Radio Engr. 35, 7, 250-251 (1958), 1-title bibl.

"Tape wound cores. Bobbin cores," Electronics, Engineering Edition 31, 19, 125 (1958).

"Toroidal coil winder," Instr. and Automation, 32, 7, 1240 (1958), 1 fig.

"Toroidal components," Electronic Industries and Tele-Tech. No. 1, p. 116 (1958).

3. General Questions in the Theory of Nonlinear Magnetic Circuits (General Methods for Computing Circuits with Steel, Ferroresonance, etc.)

S. A. Abbas and D. L. Critchlow, "Calculation of flux patterns in ferrite multipath structures," IRE Nat. Convention Rec. 6, 4, 263-267 (1958), 10 figs.

J. Kryze, "The use of the duality principle for the transfiguration of magnetic circuits," [in Czech.], Elektrotechn. Obzor 47, 3, 130-133 (1958), 19 figs., 8-title bibl.

H. W. Lord, "An equivalent circuit for transformers in which nonlinear effects are present," Proc. C.M.A. (1958), Los Angeles, AIEE, pp. 296-308 (1958), 18 figs., 1-title bibl., 1 table.

A. S. Mostafa and M. I. El-Karaksi, "Subharmonics in iron-cored reactor oscillatory circuits," Commun. and Electronics No. 35, pp. 104-113 (1958).

S. Oshima et al., "Analysis of parametrically excited circuits – parametron and magnetic amplifiers," J. Inst. Elec'r. Commun. Engrs. of Japan 41, 10, 3-5 (1958).

H. F. Storm, "Applications of nonlinear magnetics. Part I," Electronic Design 6, 4 (1958); Part II – 6, 6, 44-47, 4 figs., 12-title bibl.; Part III – 6, 8, 62-65, 5 figs. 17-title bibl.; Part IV – 6, 10, 60-63, 5 figs, 29-title bibl.

F. G. Wesby, "Electromagnetic fields in a ferromagnetic medium with particular reference to harmonic distortion due to hysteresis," Proc. IEE, 105 C, 7, 218-229 (1958).

4. Magnetic Amplifiers. Survey and Applications

a) Survey

"Control" Standard Reactors, Catalog S-10, R-10, T-10, Control– a division of Magnetics, Inc., Butler, Pennsylvania, U.S.A. (1958).

L. A. Finzi, "Introducing young engineers to the appreciation of magnetic amplifier problems," Commun. and Electronics No. 35, pp. 119-126 (1958), 8 figs, 20-title bibl.; Elec. Eng. 77, 11, 1035-1037 (1958), 1-title bibl.

M. Frank and J. R. Walker, "High-temperature magnetic amplifiers," Electronic Components Symposium (Chicago, 1957), pp. 186-193 (1957).

W. A. Geyger, " Application of magnetic amplifiers in industrial instrumentation and control," Trans. IRE, 4, pt. PGIE 5, 4, 23-36 (1958).

H. B. Harms, "Magnetic materials in ultrahigh temperature electronic transformers," Elec. Eng. 77, 5, 408-412 (1958), 12 figs, 2-title bibl.; Commun. and Electronics No. 36, pp. 181-184 (1958), 8 figs, 2-title bibl.

H. B. Harms et al. "Ultrahigh temperature (500°C) miniaturized power transformers and inductors," Proc. Electronic Components Symposium, pp. 177-185 (1957).

W. G. Harvey, "Magnetic amplifier design principles and some applications," L. S. E. Engng. Bull. 5, 1, 1-13 (1958), 22 figs, 4-title bibl.

"High stability magnetic amplifiers," Process Control and Automation 5, 11, 11 (1958).

H. Hill and F. M. Kinney, "Magnetic amplifier selection by nomograph," El. Manufacturing 62, 4, 80-81, 282, 284, 286 (1958), 2-title bibl.

J. D. Kopsho, "A synchronous power converter for high-efficiency magnetic amplifiers," Proc. STC on Mag. Amp. T-98, pp.8 (August, 1957).

A. Krausz, "Application of magnetic amplifiers in aircraft electrical systems," Proc. NEC 13, 926-933 (1957), 7 figs, 3-title bibl.

K. Kuhnert, "The role of magnetic amplifiers in industrial control" [in French], Instrum. et Labs. 5, 9-18, (1958), 20 figs.

A. Lang, "Magnetic amplifiers – reliable elements of automation" [in German], Automatik 13, 8, 201-206 (1958), figs. 14-26, 14-title bibl.

A. Lang, "New applications of magnetic amplifiers" [in German], Scientia Electr. 4, 3, 61-68 (1958).

C. K. Leonard, "High-frequency temperature-stable magnetic amplifiers," Proc. C.M.A. (1958), Los Angeles, AIEE, pp. 405-413 (1958), 11 figs.

"Low-level preact magnet amplifiers," Electronics No. 13, pp. 92 (1958).

"Magnetic amplifier low-level unit," Electronics, Engineering Edition 31, 11, 208 (1958), 1 fig.

"Magnetic amplifiers and transductors," Control 1, 4 (1958).

"Magnetic amplifiers feature variable gain," Electronics, Business Edition 31, 2, 41-42 (1958), 1 fig.

E. W. Miesner, "Magnetic amplifiers and controlling and regulating devices for electric motors" [in German], Elektrotechnik 39, 44, 361-364 (1957).

A. G. Milnes, "Transistors versus magnetic amplifiers," Power Eng. pp. 88-89 (December, 1958).

J. W. S. Payne, "The performance and application of magnetic amplifiers," Students Quart. J. 28, 112, 173-179 (1958).

L. Pun, "On the simulation of certain industrial systems by small electric motors" [in French], Automatisme 3, 12, 461-468 (1958), 15 figs.

D. A. Ramsay and B. W. Glover, "British magnetic amplifier developments," El. Manufacturing 62, 56-62 (July, 1958), 19 figs., 9-title bibl.

R. W. Spenser and T. H. Bonn, "Magnetic amplifier circuits with feedback," U.S. patent, class 307-88; No. 2812449; 11/5/1957.

L. W. Stammerjohn, "Magnetic amplifiers: basic principles and applications," Bell Labs. Rec. 36, 1, 16-21 (1958), 7 figs.

J. C. Taylor and Ch. L. Wyman, "Transistor circuits alter magnetic amplifier frequency response," El. Manufacturing 62, 2, 90-91 (1958), 8 figs., 16-title bibl.

"Transistor-magnetic amplifier for servomotor control," Electronics, Engineering Edition 31, 27, 122 (1958).

R. Zadsinski, "Universal characteristics of magnetic amplifiers " [in Polish], Archiwum automatyki i telemechaniki 3, 1-2, 17-35 (1958), 8 figs., 9-title bibl.

b) Application in Servo Systems and for Controlling Electric Machines

E. Bernard, "The control of two-phase motors by magnetic amplifiers" [in French], Automatisme 3, 2, 71-75 (1958), 7 figs, 3-title bibl.

H. C. Bourne et al., "Dynamic operation of magnetic amplifiers for feedback control systems," Electronics

Research Lab., U. of Calif., Berkeley, Scientific rept. No. 4, p. 27 (May, 1957).

K. M. Chigrwin, "Aircraft switching-type static inverter supplying rotating machines," Applic. and Ind. No. 39, pp. 334-335 (November, 1958).

H. E. Darling, "Magnetic amplifiers for process control," El. Design 29, p. 3 (October, 1958).

"A dc instrument with magnetic amplifiers" [in German], Elektrotechn. Anz. 74, 6, 142 (1957).

Mitio Fuzdzisava, "Controlling a paper-cutting machine by magnetic amplifiers," Avtomatika 3,2,54-58 (1958).

N. F. Horton and H. J. Tyler, "Magnetic amplifier everspeed trip devices (West. Electric Corp.). 5/17/1952. U.S. patent, No. 2760149; 8/21/1956.

G. B. Houck, "Transistor magnetic amplifier circuit, (General Precision Lab. Inc.), U.S. patent, class 179-171, No. 2,819,352; 1/7/1958.

Cigasi Kobajacy and Kanjitiro Morita, "A G-D device controlled by a magnetic amplifier," Japanese patent, class 55c, No. 211, No. 5011, 6/26/1956.

F. Kummel, "Stabilization of a control system with magnetic amplifiers," [in German, Regelungstechnik 6, 6, 209-217 (1958).

P. Lamorlette, "Control equipment for a dc generator" [in French], French patent, class 12-6, No. 1128660; 1/9/1957.

G. Loocke, "Problems in the control and regulation of electric drives (G-D system)" [in German], Energie 10, 5, 198-203 (1958).

H. G. Lott, "Use of magnetic amplifier for braking of 3-phase asynchronous motors," AEG Mitt. 47, 1-2, 19-23 (1957).

R. Mathieson, "Electric drives at consett," Control 1, 5, 235-239 (1958), 7 figs.

V. Nemec, "Synchronous generators with automatic contactless control," [in Czech.], Elektrotechnik 12, 7 225-228 (1957), 11 figs, 5-title bibl.

E. M. Ostaseski, "Amplistats in control," Automatic Control 6, 4, 41-43 (1957), 3 figs.

K. Raedle, "The use of rectifiers and magnetic amplifiers in metallurgy" [in French], Bull. Sci. AIM 71, 2, 89-116 (1958).

S. J. Reisman and E. H. Bowman, "A power method of measuring the torque-producing output of a magnetic amplifier driving one phase of a two-phase servo motor," Commun. and Electronics No. 39, pp. 584-586 (1958), 3 figs.

A. Straughen, P. P. Biringer, and G. R. Slemon, "Three-phase induction motor control using static-frequency doublers," Commun. and Electronics No. 36 (1958).

N. E. Strom, "The speed regulation of small dc machines by means of transductor-controlled rectifiers," ASEA Journal 31, 3, 40-43 (1958), 6 figs.

M. E. Wright, "Press power drive system at Philadelphia Bulletin," Print. Equipm. Engr. 87, 9, 76-80, 112, 113 (1957), 4 figs.

E. Zucker, "Unique ac hoists speed production," El. Digest 27, 3, 55-56, 58 (1958).

c) Use in Controllers and Stabilizers of Current and Voltage

"A complete line of magnetic voltage regulators," Electronics 31, 29 (1958).

W. E. Cronquist, "A simple high-performance magnetic amplifier generator voltage regulator," Proc. STC on Mag. Amp. T-98, p. 245 (August, 1957).

I. K. Dortort, "Current-balancing reactors for semiconductor rectifiers," Commun. and Electronics No. 38, pp. 452-456 (1958).

L. Gador, "Transformer voltage regulator" [in Czech.], Meres es Automatika No. 1, 1-12, (1958), 5 figs.

H. V. Hjermstad and J. T. Keefe, "Static-magnetic regulated dc power supplies," El. Manufacturing 62, 110-113, 270 (1958).

O. Hora, "A Ferroresonant voltage stabilizer" [in Czech], Automatisace No. 5, 157-162 (1958), 15 figs, 7-title bibl.

M. A. Johansson, "A new supply-voltage regulator of transductor type," ASEA Journal 31, 3, 37-40 (1958), 6 figs.

K. King, "Improvements relating to electric control apparatus," English patent, class 40/4, No. 752340; 7/11/1956.

"A magnetic amplifier-regulator" [in German], Regelungstechnik 6, 7, 272-273 (1958), 3 figs.

M. J. Mulhern, "Regulating circuit for energizing variable resistance loads," U.S. Patent, class 315-106, No. 2822507; 2/4/1958.

d) Use in Measuring Devices

H. E. Darling, "DC transformer has continuous adjustable ratio," El. Des. 5, p. 4 (March, 1958).

W. A. Geyger, "Magnetic-amplifier-operated ink recorders," Commun. and Electronics No. 38, pp. 457-471 (1958), 43-title bibl.

A. R. Hamilton, "A magnetic amplifier control circuit for a thermal conductivity vacuum gauge," 4th Mat. Sympos. on Vacuum Technology Trans. (Boston, 1957), pp. 112-114 (1958).

J. McFarlane and M. J. Harris, "The control of flux waveforms in iron testing by the application of feedback amplifier techniques," Proc. IEE 8, 22, 105A, 395-405 (1958).

H. W. Patton, "Magnetic integrating amplifiers. Their design and application," Proc. C.M.A. (1958), Los Angeles, AIEE, pp. 267-277 (1958), 9 figs. 5-title bibl.

A. S. Watson, "An improved transductor dc ammeter," El. Energy 2, 7, 275-281 (1958).

E. V. Weir, "Magnetic amplifier circuit using complex feedback," U.S. patent, class 323-89, No. 2817807; 12/24/1957.

e) Other Applications

A. O. Adams, "Magnetic-amplifier-operated relays for aircraft and missiles," Electronic Industries 17, 12, 12, 72-74, 153 (1958).

P. W. Covert and E. V. Weir, "A transductor summation system for strip plater," Proc. NEC 13, 934-942 (1957), 8 figs., 3-title bibl.

"New stepping switches for power transformers based on the operational principle of a magnetic amplifier" [in German] ETZ-A 79, 23, 934 (1958).

C. L. Hogan, "Magnetoelectric induction devices," U.S. patent, class 333-32, No. 2811697; 19/ 29/ 1957.

O. Hora, "The use of heat-sensitive chokes and transformers" [in Czech], Automatisace No. 3, 78-82 (1958), 10 figs., 6-title bibl.

R. V. Hubner, "The use of saturated reactors for controlling mercury rectifiers," [in German], ETZ-A 79, 15, 530-532 (1958), 5 figs.

W. E. Jewett and P. L. Schmidt, "A more stable three-phase transistor-core power inverter," Proc. C.M.A. (1958) Los Angeles, AIEE, pp. 348-362 (1958), 11 figs., 4-title bibl.

J. Maroz, "Magnetic amplifier detects open fuses," Electronics 31, 29, 86- 87 (1958), 1 fig.

B. A. Mazzeo, "A low-level, high accuracy dc magnetic amplifier," El. Manufacturing 62, 5, 106-110 (1958), 12 figs.

W. F. Nesbitt, "Obtaining optimum performance from a magnetic thermocouple amplifier, " Trans. IRE, PGIE-6, pp. 101-114 (1958).

"New annunciator system for generating stations," Westinghouse Engr. No. 1, p. 11 (1958).

A. Newton, "Saturable variable inductor," U.S. patent, class 336-155, No. 2826748; 3/ 11/ 1958.

D. L. Plette and J. W. Butler, "Some applications of magnetic amplifiers in aircraft protective systems," Applic. and Ind. 34, 427-433 (January, 1958), 16 figs. 2-title bibl.

R. P. Putkovich and T.M. Corry, "Application of saturable core in transistor power converters," Proc. NEC 14, 874-883 (1958), 10 figs., 7-title bibl.

A. Rose, "Magnetic-core dividers for I.T.V. (Industrial television) sync generators," Electronics 31, 15, 76-77 (1958).

N. R. Schuh, "Frequency-sensitive control device," West. Electrical Corp., 10/ 15/ 1953; U. S. patent, No. 2757320; 6/ 31/ 1956.

H. E. Thomas, "Saturable reactors fire radar magnetrons," Electronics 31, 19, 72-75 (1958).

P. K. Tien and H. A. Suhl, "Traveling-wave ferromagnetic amplifier," Proc. IRE 46, 700-706 (April, 1958).

C. C. Voice, "Magnetic amplifier drives gyro indicator," Electronics 31, 7, 114-117 (1958), 6 figs., 4-title bibl.

5. Magnetic Amplifiers. Theory, Circuits, Calculations

a) Books, Monographs, Dissertations

G. M. Ettinger, Magnetic Amplifiers (Methuen and Co., London, 1957) issue 2, p. 102.

E. H. Frost-Smith, "The Theory and Design of Magnetic Amplifiers (Chapman and Hall, Ltd., London, 1958) p. 487, 151 figs., 57-title bibl.

S. Platt, Magnetic Amplifiers. Theory and Application (Prentice-Hall, Inc. Englewood Cliffs, N. J., 1958) p. 238.

b) Nonsequential Magnetic Amplifiers (Single-Cycle)

R. E. Anderson, "Magnetic amplifier design. I. The core. II. The windings," Commun. and Electronics No. 36, pp. 160-175 (May, 1958); El. Engng. 77, 5, 394-395 (1958), 3 figs.

E. Bernard and E. Pio, "A contribution to the study of magnetic amplifier operation," [in French], C.R. Acad. Sc. 248, 394-397 (1959).

H. H. Britten, "Characteristics of magnetic amplifiers utilizing square-loop core material and square-wave supply voltage," Commun. and Electronics No. 39, pp. 590-598 (1958), 11 figs., 3-title bibl., discussion.

L. W. Buechler and W. F. Schmidt, "Magnetic amplifier system," U.S. patent, class 323-89, No. 2807776; 9/ 24/ 1957.

D. P. Chandler and R. W. Downing, "Elimination of magnetic-amplifier control circuit inductance," Commun. and Electronics No. 39, pp. 550-556 (November, 1958), 18 figs., 1 table, 11-title bibl., disc. pp. 556-558, 4-title bibl.; Proc. STC on Magn. Amp. T-98, p. 21 (August, 1957).

E. L. Clarke "Auto. self-excited transductors. A nonmathematical approach," Instrument Practice 12, 10, 1093-1100 (1958), 20 figs., 2-title bibl.

C. Delgrosso and G. Sacco , "Methods of modifying the dynamic response of magnetic amplifiers," Electrotecnica 44, 6, 318-330 (June, 1957).

J. P. Eckert, "Bipolar output carrier magnetic amplifier," U.S. patent, class 307-88, No. 2808520; 10/ 1/ 1957.

J. P. Eckert and T. H. Bonn, "Carrier-type magnetic amplifier with a feedback circuit," U. S. patent, class 307-88, No. 2830198; 4/ 8/ 1958.

G. M. Ettinger, "Some aspects of half-wave magnetic amplifiers," Proc. IRE 105B, 21, 237-248 (1958).

E. H. Frost-Smith, "Fast-response magnetic amplifiers," Control 1, 4, 176-182 (1958), 9 figs., 7-title bibl.

E. L. Gardner, "Magnetic amplifier," U.S. patent, class 179-171, No. 2812391; 11/ 5/ 1957.

B. W. Glover, "Magnetic amplifiers with half-cycle response," El. Energy 2, 1, 2-9 (1958); 2, 2, 58-64 (1958).

J. C. R. Heydenrych, "A note on the design of transductors for maximum power transfer," Trans. S. African Inst. Elec. Engrs. 48, pt. 12, 370-377 (1957).

R. M. Hubbard, "Block-diagram method of analysis applied to the saturable reactor," Commun. and Electronics, No. 35, pp. 57-65 (March, 1958).

R. M. Hubbard and M. M. Bishop, "A comprehensive study of magnetic amplifier design. Part I. A practical design method. Part II. Toroidal-core dimensional ratios as a factor in magnetic amplifier design," Proc. STC on Mag. Amp. T-98, p. 121 (August, 1957); Commun. and Electronics No. 39, p. 562-583, 15 figs., 3 tables, 3-title bibl.

W. Jentsch, "Average value of magnetic flux and inertia of magnetic amplifiers" [in German], Regelungstechnik 6, 1, 9-14 (1958).

R. H. Kelsall, J. Sherlock, and J. M. Foye, "Improvements relating to saturable core reactance devices," English patent, class 40(4), No. 753356; 7/26/1956.

E. Levi, "New amplifiers for automatic control of active dc leads," IRE Nat. Convention Rec. 6, 6, 216-222 (1958).

M. Mamon, "High speed magnetic amplifiers," Electr. Design News. 3, 9, 48-51 (1958), 5 figs., 3-title bibl.

W. L. Marks, "Bistable Ramey magnetic amplifiers," El. Energy 2, 6, 218-221 (1958).

D. L. Mc Murtrie, "Magnetic amplifier circuits – a classification of half-wave and full-wave nonreversible and reversible self-saturating circuits," Proc. C.M.A. (1958), Los Angeles, AIEE, pp. 67-86 (1958), 22 figs., 4-title bibl., 3 tables.

R. E. Morgan, "A new control amplifier using a saturable current transformer and a switching transistor," Commun. and Electronics No. 39, pp. 557-562 (November, 1958), 15 figs., 4-title bibl., disc.

J. Nagy, "Basic principles in designing magnetic amplifiers" [in Czech.], Meres es Automatika No. 1, 6-9 (1958), 7 figs., 3-title bibl.

K.C. Parton, "Improvements in or relating to magnetic amplifier arrangements," English patent, class 40/(No. 753363; 7/25/1956.

E. Pio, "Investigation of magnetic amplifiers with self-saturation" [in French], Automatisme 3 (1958), in five parts: I: No. 2, pp. 63-70, 18 figs.; II: No. 4, pp. 144-149, 32 figs.; III: No. 6, pp. 218-244, 50 figs.; IV: No. 8, pp 293-298, 58 figs.; V: No. 11, pp. 430-438, 7[illegible] figs., 11-title bibl.

T. J. Pula, "Volt-second transfer efficiency in fast response magnetic amplifiers. Part I. No. 2, Reg. and control," Proc. STC on Mag. Amp. T-98, p. 107 (August, 1957).

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